Functional analysis – part 14

Example: $X = (C([0,1], \mathbb{F}), \|\cdot\|_\infty)$, $Y = (\mathbb{F}, |\cdot|)$

For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define

$$T_g : X \longrightarrow Y \quad \text{by} \quad T_g(f) := \int_0^1 g(t) \cdot f(t)\, dt$$

What is $\|T_g\|$?

$$\|T_g\| = \sup \left\{ \frac{|T_g(f)|}{\|f\|_\infty} \;\middle|\; f \in X,\ f \neq 0 \right\}$$

$$= \sup \left\{ |T_g(f)| \;\middle|\; f \in X,\ \|f\|_\infty = 1 \right\}$$

$$= \sup \left\{ \underbrace{\left| \int_0^1 g(t) \cdot f(t)\, dt \right|}_{\leq \int_0^1 |g(t)| \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty = 1} dt} \;\middle|\; f \in X,\ \|f\|_\infty = 1 \right\}$$

$$\leq \int_0^1 |g(t)|\, dt < \infty$$

Check the other inequality: $h(t) := \dfrac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_\infty = 1$

$$\|T_g\| \geq |T_g(h)| = \left| \int_0^1 g(t) \frac{\overline{g(t)}}{|g(t)|}\, dt \right| = \int_0^1 \frac{|g(t)|^{\cancel{2}}}{\cancel{|g(t)|}}\, dt = \int_0^1 |g(t)|\, dt$$