

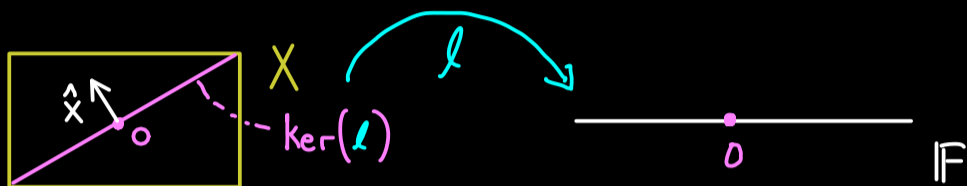
Functional analysis - part 15

Riesz representation theorem

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for each continuous linear map $l: X \rightarrow \mathbb{F}$ (a continuous linear functional) there is exactly one $x_l \in X$ such that $l(x) = \langle x_l, x \rangle$ for all $x \in X$ and $\|l\|_{X \rightarrow \mathbb{F}} = \|x_l\|_X$.

[In physics $l = \langle \gamma |$]

Proof: (1) Existence:



First case: $\ker(l) = X \Rightarrow x_l = 0$

Second case: $\ker(l) \neq X \rightsquigarrow x_l \in \ker(l)^\perp \cong \{0\}$ true because $\ker(l)$ is closed and "orthogonal projections" exist in Hilbert spaces (later)
 Kernel is preimage of closed set $\{0\}$
 continuity \rightsquigarrow Kernel is closed.

Choose $\hat{x} \in \ker(l)^\perp$ with $\|\hat{x}\|_X = 1$. Set $x_l := \overline{l(\hat{x})} \cdot \hat{x}$

$$\begin{aligned} \underline{l(x)} &= l\left(x - \frac{l(x)}{l(\hat{x})} \hat{x} + \frac{l(x)}{l(\hat{x})} \hat{x}\right) = \underbrace{l\left(x - \frac{l(x)}{l(\hat{x})} \hat{x}\right)}_{l(x) - \frac{l(x)}{l(\hat{x})} l(\hat{x}) = 0} + \underbrace{l\left(\frac{l(x)}{l(\hat{x})} \hat{x}\right)}_{\lambda} \\ &= \lambda \cdot l(\hat{x}) \cdot \langle \hat{x}, \hat{x} \rangle = \lambda \cdot \langle \overline{l(\hat{x})} \hat{x}, \hat{x} \rangle = \langle x_l, \lambda \hat{x} \rangle \\ &= \langle x_l, \lambda \hat{x} - x + x \rangle = \langle x_l, x \rangle \end{aligned}$$

(2) Uniqueness: Assume $x_l, \tilde{x}_l \in X$ fulfil $l(x) = \langle x_l, x \rangle = \langle \tilde{x}_l, x \rangle$

$$\Rightarrow \langle x_l - \tilde{x}_l, x \rangle = 0 \text{ for all } x \in X.$$

$$\Rightarrow \langle x_l - \tilde{x}_l, x_l - \tilde{x}_l \rangle = 0 \Rightarrow x_l = \tilde{x}_l$$

(3) Operator norm:

$$\begin{aligned} \|l\| &= \sup \{ |l(x)| \mid \|x\|_X = 1 \} = \sup \{ |\langle x_l, x \rangle| \mid \|x\|_X = 1 \} \\ &\leq \|x_l\| \end{aligned}$$

$$\|l\| \geq \left| l\left(\frac{x_l}{\|x_l\|}\right) \right| = \left| \langle x_l, \frac{x_l}{\|x_l\|} \rangle \right| = \|x_l\| \quad \square$$