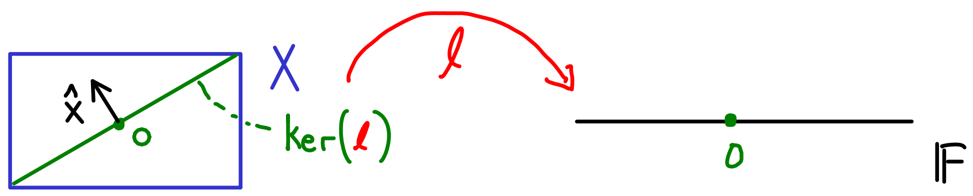
Functional analysis – part 15

Riesz representation theorem

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then for each continuous linear map $\ell : X \to \mathbb{F}$ (a continuous linear functional) there is exactly one $x_\ell \in X$ such that $\ell(x) = \langle x_\ell, x\rangle$ for all $x \in X$ and $\|\ell\|_{X\to\mathbb{F}} = \|x_\ell\|_X$.

[In physics $\ell = \langle \psi |$]

Proof: (1) Existence:



First case: $\ker(\ell) = X \Rightarrow x_\ell = 0$

Second case: $\ker(\ell) \neq X \rightsquigarrow x_\ell \in \underbrace{\ker(\ell)^\perp}_{\supsetneqq \{0\}}$ — true because $\ker(\ell)$ is closed and "orthogonal projections" exist in Hilbert spaces → later

Kernel is preimage of closed set $\{0\}$ $\overset{\text{continuity}}{\rightsquigarrow}$ Kernel is closed.

Choose $\hat{x} \in \ker(\ell)^\perp$ with $\|\hat{x}\|_X = 1$. Set $x_\ell := \overline{\ell(\hat{x})}\cdot\hat{x}$

$$\ell(x) = \ell\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x} + \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right) = \ell\underbrace{\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right)}_{\ell(x) - \frac{\ell(x)}{\cancel{\ell(\hat{x})}}\cdot\cancel{\ell(\hat{x})}\,=\,0} + \ell\left(\underbrace{\frac{\ell(x)}{\ell(\hat{x})}}_{\lambda}\hat{x}\right)$$

$$= \lambda\cdot\ell(\hat{x})\cdot\langle\hat{x},\hat{x}\rangle = \lambda\cdot\langle\overline{\ell(\hat{x})}\hat{x}, \hat{x}\rangle = \langle x_\ell, \lambda\hat{x}\rangle$$

$$= \langle x_\ell, \underbrace{\lambda\hat{x} - x}_{\in \ker(\ell)} + x\rangle = \langle x_\ell, x\rangle$$

(2) Uniqueness: Assume $x_\ell, \tilde{x}_\ell \in X$ fulfil $\ell(x) = \langle x_\ell, x\rangle = \langle\tilde{x}_\ell, x\rangle$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x\rangle = 0 \quad \text{for all } x \in X.$$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x_\ell - \tilde{x}_\ell\rangle = 0 \Rightarrow x_\ell = \tilde{x}_\ell$$

(3) Operator norm:

$$\|\ell\| = \sup\left\{|\ell(x)| \;\middle|\; \|x\|_X = 1\right\} = \sup\left\{\underbrace{|\langle x_\ell, x\rangle|}_{\leq \|x_\ell\|_X\cdot\|x\|_X} \;\middle|\; \|x\|_X = 1\right\}$$

$$\leq \|x_\ell\|$$

$$\|\ell\| \geq \left|\ell\left(\frac{x_\ell}{\|x_\ell\|}\right)\right| = \left|\left\langle x_\ell, \frac{x_\ell}{\|x_\ell\|}\right\rangle\right| = \|x_\ell\| \qquad \square$$