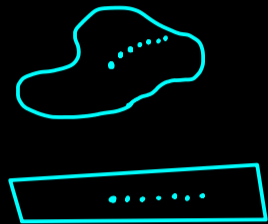


Functional analysis - part 16

Compactness $\mathbb{R}^n \supseteq A$

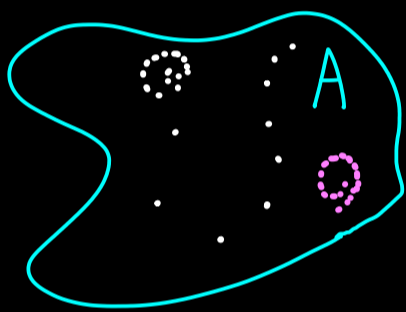
A is compact = $\begin{cases} \bullet A \text{ is closed} \\ \bullet A \text{ is bounded} \end{cases}$
only in \mathbb{R}^n or \mathbb{C}^n



Definition:

Let (X, d) be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ one finds a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with

$$\tilde{x} := \lim_{k \rightarrow \infty} x_{n_k} \in A$$



Examples:

- (a) $(\mathbb{R}, d_{\text{eucl.}})$, $A = [0, 1]$ compact by Bolzano-Weierstrass theorem.
(b) $(\mathbb{R}, d_{\text{discr.}})$, $A = [0, 1]$ not compact because:

The sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies

$$d_{\text{discr.}}(x_n, x_m) = 1 \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m.$$

\Rightarrow no convergent subsequence

Proposition:

Let (X, d) be a metric space and $A \subseteq X$ compact.

Then A is closed and bounded. There is an $x \in X$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \supseteq A$



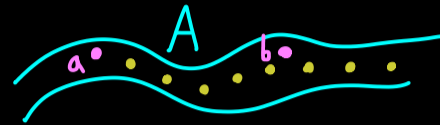
Proof: Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n \in \mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$\stackrel{\text{compact}}{\Rightarrow}$ There is a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with limit $\tilde{\tilde{x}} \in A$

$\stackrel{\text{limit unique}}{\Rightarrow} \tilde{x} = \tilde{\tilde{x}} \in A \Rightarrow A$ is closed

(2) Contraposition: A is not bounded



\Rightarrow For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

\Rightarrow For any subsequence $(x_{n_k})_{k \in \mathbb{N}}$ and any point $b \in A$:

$$n_k < d(a, x_{n_k}) \leq d(a, b) + d(b, x_{n_k})$$

$$\Rightarrow n_k - d(a, b) \leq d(b, x_{n_k})$$

$\Rightarrow d(b, x_{n_k}) \not\xrightarrow{k \rightarrow \infty} 0$ for all $b \in A \Rightarrow A$ not compact