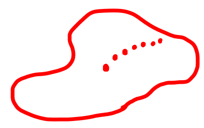
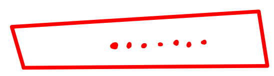
Functional analysis – part 16

Compactness $\quad \mathbb{R}^n \supseteq A$

$$A \text{ is compact} = \begin{cases} \bullet\ A \text{ is closed} \\ \bullet\ A \text{ is bounded} \end{cases}$$

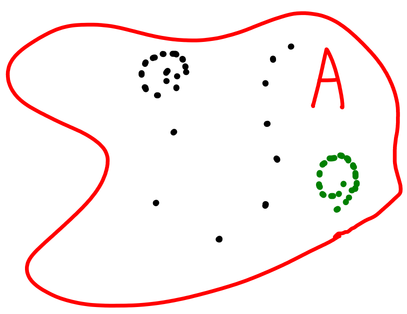




only in $\mathbb{R}^n$ or $\mathbb{C}^n$

Definition: Let (X,d) be a metric space. $A \subseteq X$ is called **(sequentially) compact** if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ one finds a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$\tilde{x} := \lim_{k\to\infty} x_{n_k} \in A$$



Examples:

(a) $(\mathbb{R}, d_{\text{eucl.}})$, $A = [0,1]$ compact by Bolzano-Weierstrass theorem.

(b) $(\mathbb{R}, d_{\text{discr.}})$, $A = [0,1]$ not compact because:

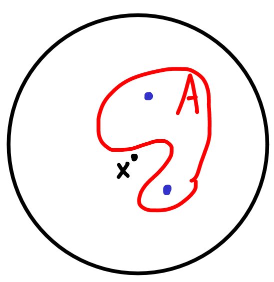
The sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies

$$d_{\text{discr.}}(x_n, x_m) = 1 \quad \text{for all } n,m \in \mathbb{N} \text{ with } n \neq m.$$

$\Rightarrow$ no convergent subsequence

Proposition: Let (X,d) be a metric space and $A \subseteq X$ compact. Then A is closed and bounded.

There is an $x \in X$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \supseteq A$



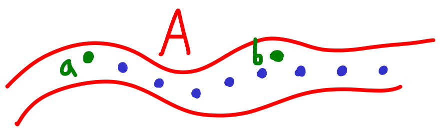
Proof: Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$\overset{\text{compact}}{\Longrightarrow}$ There is a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $\tilde{\tilde{x}} \in A$

$\overset{\text{limit unique}}{\Longrightarrow}$ $\tilde{x} = \tilde{\tilde{x}} \in A \quad \Rightarrow \quad A$ is closed

(2) Contraposition: A is not bounded



$\Rightarrow$ For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

$\Rightarrow$ For any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and any point $b \in A$:

$$n_k < d(a, x_{n_k}) \leq d(a, b) + d(b, x_{n_k})$$

$$\Rightarrow \quad n_k - d(a, b) \leq d(b, x_{n_k})$$

$\Rightarrow$ $d(b, x_{n_k}) \overset{k\to\infty}{\not\longrightarrow} 0$ for all $b \in A$ $\quad\Rightarrow\quad A$ not compact