

Functional analysis - part 20

Minkowski's inequality: Δ -inequality for $\|\cdot\|_p$ in $\ell^p(\mathbb{N})$:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \text{for all } x, y \in \ell^p(\mathbb{N}), \quad p \in [1, \infty)$$

Proof: For $p=1$: $\|x+y\|_1 = \sum_{j=1}^{\infty} \underbrace{|x_j+y_j|}_{\leq |x_j|+|y_j|} \leq \|x\|_1 + \|y\|_1$

For $p \in (1, \infty)$: Hölder conjugate $p' \in (1, \infty)$

$$\boxed{\frac{1}{p} + \frac{1}{p'} = 1}$$

$$\frac{p}{p-1} = p'$$

$$\|x+y\|_p^p = \sum_{j=1}^{\infty} |x_j+y_j|^p = \lim_{n \rightarrow \infty} \sum_{j=1}^n \underbrace{|x_j+y_j|^p}_{\leq (|x_j|+|y_j|)^p} = (*)$$

$$(**) (|x_j|+|y_j|)^p = (|x_j|+|y_j|) (|x_j|+|y_j|)^{p-1} = \underbrace{|x_j|}_{a_j} \underbrace{(|x_j|+|y_j|)^{p-1}}_{b_j} + \underbrace{|y_j|}_{c_j} (|x_j|+|y_j|)^{p-1}$$

$a_j, b_j, c_j \leadsto a, b, c \in \mathbb{F}^n$

Hölder: $\|ab\|_1 \leq \|a\|_p \cdot \|b\|_{p'}$

$$= \left(\sum_{j=1}^n |(|x_j|+|y_j|)^{p-1}| \right)^{\frac{1}{p'}} = \left(\sum_{j=1}^n (|x_j|+|y_j|)^p \right)^{\frac{1}{p'}}$$

$$(***) \sum_{j=1}^n (|x_j|+|y_j|)^p \leq \|a\|_p \cdot \|b\|_{p'} + \|c\|_p \cdot \|b\|_{p'} = (\|a\|_p + \|c\|_p) \cdot \left(\sum_{j=1}^n (|x_j|+|y_j|)^p \right)^{\frac{1}{p'}}$$

$$\Rightarrow \left(\sum_{j=1}^n (|x_j|+|y_j|)^p \right)^{1-\frac{1}{p'}} \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}}$$

$$\xrightarrow[+ (*)]{n \rightarrow \infty} \|x+y\|_p \leq \|x\|_p + \|y\|_p$$