

Functional analysis - part 23

Dual space: X normed space

$$X' := \left\{ \ell: X \rightarrow \mathbb{F} \mid \ell \text{ linear + bounded} \right\}$$

Example: $X = \ell^p(\mathbb{N})$ for $p \in (1, \infty)$

$$X' \cong \ell^{p'}(\mathbb{N}) \quad \text{where } p' \in (1, \infty) \text{ Hölder conjugate } \left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$$

there is an isometric isomorphism

$$T: \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'$$

$$(Tx)(y) := \sum_{k=1}^{\infty} x_k \cdot y_k \quad \text{or } x \mapsto \langle \bar{x}, \cdot \rangle_{\ell^p(\mathbb{N})}$$

To show: (1) T is well-defined ✓

(4) T surjective

(2) T is linear ✓

(5) $\|Tx\| = \|x\|$ for all $x \in \ell^{p'}(\mathbb{N})$

(3) T is bounded ✓

(isometric)

Proof: (1) $| (Tx)(y) | \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n |y_k \cdot x_k| \stackrel{\text{Hölder}}{\leq} \|y\|_p \cdot \|x\|_{p'} < \infty$

$\Rightarrow Tx$ is linear and bounded for all $x \in \ell^{p'}(\mathbb{N})$

(2) T is linear.

$$(3) \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} = \sup \left\{ \underbrace{|(Tx)(y)|}_{\leq \|y\|_p \cdot \|x\|_{p'}} \mid \|y\|_p = 1 \right\} \leq \|x\|_{p'}$$

$$T: \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'$$

$$\Rightarrow \|T\| \leq 1$$

(4) Let $\gamma' \in (\ell^p(\mathbb{N}))'$ and $e_k = (0, 0, \dots, 0, 1, 0, \dots)$. ↖ k^{th} position

Define: $x_k := \gamma'(e_k)$ and $x := (x_k)_{k \in \mathbb{N}}$

Question: $x \in \ell^{p'}(\mathbb{N})$ and $Tx = \gamma'$?

$$\sum_{k=1}^n |x_k|^{p'} = \sum_{k=1}^n x_k \cdot t_k \begin{cases} \frac{|x_k|^{p'}}{x_k}, & x_k \neq 0 \\ 0, & x_k = 0 \end{cases}$$

$$= \sum_{k=1}^n t_k \cdot \gamma'(e_k) = \gamma'\left(\sum_{k=1}^n t_k e_k\right)$$

$$\leq \|\gamma'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot \left\| \sum_{k=1}^n t_k e_k \right\|_p = (t_1, t_2, \dots, t_n, 0, 0, \dots)$$

$$\leq \|\gamma'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \cdot \left(\sum_{k=1}^n |t_k|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^n \left(\frac{|x_k|^{p'}}{|x_k|} \right)^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^n |x_k|^{(p'-1)p} \right)^{\frac{1}{p}} = \left(\sum_{k=1}^n |x_k|^{p'} \right)^{\frac{1}{p}}$$

$$\left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$$

$$\stackrel{n \rightarrow \infty}{\Rightarrow} \|x\|_{p'} \leq \|\gamma'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \Rightarrow x \in \ell^{p'}(\mathbb{N}) \checkmark$$

$$\text{For } \gamma \in \ell^p(\mathbb{N}): (Tx - \gamma')(y) = (Tx - \gamma')\left(\lim_{n \rightarrow \infty} \sum_{k=1}^n \gamma_k e_k\right)$$

$$\stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} (Tx - \gamma')\left(\sum_{k=1}^n \gamma_k e_k\right)$$

$$\stackrel{\text{linearity}}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n \gamma_k (Tx - \gamma')(e_k) = 0 \quad \text{surjective} \checkmark$$

$$(Tx)(y) := \sum_{j=1}^{\infty} x_j \cdot \gamma_j$$

$$(5) \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \stackrel{=}{=} \|x\|_{p'} \stackrel{=}{=} \|\gamma'\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} = \|Tx\|_{\ell^p(\mathbb{N}) \rightarrow \mathbb{F}} \quad \text{isometry} \checkmark$$