Functional analysis – part 23

Dual space: X normed space

$$X' := \{ \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \}$$

Example: $X = \ell^p(\mathbb{N})$ for $p \in (1, \infty)$

$X' \cong \ell^{p'}(\mathbb{N})$ where $p' \in (1,\infty)$ Hölder conjugate $\left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$

there is an isometric isomorphism

$$T : \ell^{p'}(\mathbb{N}) \longrightarrow \left(\ell^p(\mathbb{N})\right)'$$

$$(Tx)(y) := \sum_{k=1}^{\infty} x_k \cdot y_k \qquad \text{or} \qquad x \mapsto \langle \overline{x}, \cdot \rangle_{\ell^2(\mathbb{N})}$$

To show:
(1) T is well-defined ✓
(2) T is linear ✓
(3) T is bounded ✓
(4) T surjective
(5) $\|Tx\| = \|x\|$ for all $x \in \ell^{p'}(\mathbb{N})$ (isometric)

Proof: (1) $|(Tx)(y)| \leq \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} |y_k \cdot x_k| \overset{\text{Hölder}}{\leq} \|y\|_p \cdot \|x\|_{p'} < \infty$

$\Rightarrow$ Tx is linear and bounded for all $x \in \ell^{p'}(\mathbb{N})$

(2) T is linear.

(3) $\|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} = \sup \{ \underbrace{|(Tx)(y)|}_{\leq \|y\|_p \cdot \|x\|_{p'}} \mid \|y\|_p = 1 \} \leq \|x\|_{p'}$

$$\boxed{T: \ell^{p'}(\mathbb{N}) \longrightarrow \left(\ell^{p}(\mathbb{N})\right)'}$$

$$\Rightarrow \quad \|T\| \leq 1$$

(4) Let $y' \in \left(\ell^p(\mathbb{N})\right)'$ and $e_k = (0,0,\dots,0,\underset{}{\overset{k^{th}\text{ position}}{1}},0,\dots)$.

Define: $x_k := y'(e_k)$ and $x := (x_k)_{k\in\mathbb{N}}$

Question: $x \in \ell^{p'}(\mathbb{N})$ and $Tx = y'$?

$$\sum_{k=1}^{n} |x_k|^{p'} = \sum_{k=1}^{n} x_k \cdot t_k, \qquad t_k := \begin{cases} \dfrac{|x_k|^{p'}}{x_k}, & x_k \neq 0 \\ 0, & x_k = 0 \end{cases}$$

$$= \sum_{k=1}^{n} t_k \cdot y'(e_k) = y'\Big(\sum_{k=1}^{n} t_k e_k\Big)$$

$$\leq \|y'\|_{\ell^p(\mathbb{N})\to\mathbb{F}} \cdot \underbrace{\Big\|\overbrace{\sum_{k=1}^{n} t_k e_k}^{=(t_1,t_2,\dots,t_n,0,0,\dots)}\Big\|_p}_{\left(\sum_{k=1}^{n} \underbrace{|t_k|^p}_{=\left(\frac{|x_k|^{p'}}{|x_k|}\right)^p = |x_k|^{\overbrace{(p'-1)\cdot p}^{p'}}}\right)^{\frac{1}{p}}} \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

$$= \|y'\|_{\ell^p(\mathbb{N})\to\mathbb{F}} \left(\sum_{k=1}^{n} |x_k|^{p'}\right)^{\frac{1}{p}}$$

$$\overset{n\to\infty}{\Longrightarrow} \quad \|x\|_{p'} \leq \|y'\|_{\ell^p(\mathbb{N})\to\mathbb{F}} \quad \Rightarrow \quad x \in \ell^{p'}(\mathbb{N}) \ ✓$$

For $y \in \ell^p(\mathbb{N})$: $(Tx - y')(y) = (Tx - y')\Big(\lim_{n\to\infty} \sum_{k=1}^{n} y_k e_k\Big)$

$$\boxed{(Tx)(y) := \sum_{j=1}^{\infty} x_j \cdot y_j}$$

$$\overset{\text{continuity}}{=} \lim_{n\to\infty} (Tx - y')\Big(\sum_{k=1}^{n} y_k e_k\Big)$$

$$\overset{\text{linearity}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} y_k \cdot (Tx - y')(e_k) = 0 \qquad \text{surjective ✓}$$

(5) $\|Tx\|_{\ell^p(\mathbb{N})\to\mathbb{F}} \underset{=}{\leq} \|x\|_{p'} \underset{=}{\leq} \|y'\|_{\ell^p(\mathbb{N})\to\mathbb{F}} = \|Tx\|_{\ell^p(\mathbb{N})\to\mathbb{F}}$ isometry ✓