The Bright Side of Mathematics

Functional analysis – part 24

Uniform boundedness principle (Banach-Steinhaus theorem)

X, Y normed spaces, X Banach space.

$$\mathcal{B}(X,Y) := \{ T : X \to Y \mid T \text{ linear + bounded} \}$$

Theorem: For every subset $\mathcal{M} \subseteq \mathcal{B}(X,Y)$ holds:

$\mathcal{M}$ is bounded pointwise on X $\iff$ $\mathcal{M}$ is uniformely bounded

More concretely: $\underset{x \in X}{\forall} \; \underset{C_x \geq 0}{\exists} \; \underset{T \in \mathcal{M}}{\forall} \; \|Tx\|_Y \leq C_x \iff \underset{C \geq 0}{\exists} \; \underset{T \in \mathcal{M}}{\forall} \; \|T\|_{X \to Y} \leq C$

Proposition: X, Y normed spaces, X Banach space.

Let $T_n \in \mathcal{B}(X,Y)$ for all $n \in \mathbb{N}$ with $\lim_{n \to \infty} T_n x$ exists for all $x \in X$.

Then: $T : X \to Y$ defined by $Tx := \lim_{n \to \infty} T_n x$ is linear and bounded.

Proof: $\mathcal{M} := \{ T_n \mid n \in \mathbb{N} \}$ is bounded pointwise on X $\overset{\text{Banach–Steinhaus}}{\Longrightarrow}$ There is a $C \geq 0$ with $\|T_n\| \leq C$ for all n

$$\Rightarrow \|T\|_{X \to Y} = \sup \{ \underbrace{\|Tx\|_Y} \mid \|x\|_X = 1 \} \leq C$$

$$\| \lim_{n \to \infty} T_n x \|_Y = \lim_{n \to \infty} \|T_n x\|_Y \leq C$$