Functional analysis – part 25

Hahn–Banach theorem $(X, \|\cdot\|_X)$ normed space $\rightsquigarrow$ $(X', \|\cdot\|_{X'})$

$U \subseteq X$ subspace, $u' : U \longrightarrow \mathbb{F}$ continuous linear functional

Then: There exists $x' : X \longrightarrow \mathbb{F}$ continuous linear functional

with $x'(u) = u'(u)$ for all $u \in U$,

$\|x'\|_{X'} = \|u'\|_{U'}$.

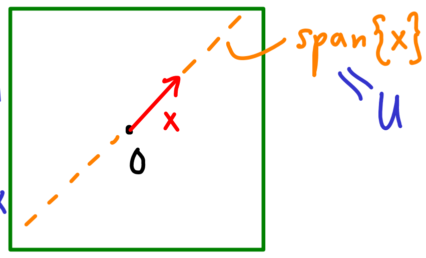
Applications: $(X, \|\cdot\|_X)$ normed space

(a) For all $x \in X$, $x \neq 0$, there is an $x' \in X'$ with $\|x'\|_{X'} = 1$ and $x'(x) = \|x\|_X$.

Proof: Define $u' : U \longrightarrow \mathbb{F}$, $\lambda \cdot x \longmapsto \lambda \cdot \|x\|_X$ continuous linear functional

$\overset{\text{Hahn–Banach}}{\Longrightarrow}$ $x' : X \longrightarrow \mathbb{F}$ with $x'(x) = u'(x) = \|x\|_X$

$\|x'\|_{X'} = \|u'\|_{U'} = 1$



(b) X' separates the points of X: For $x_1, x_2 \in X$, $x_1 \neq x_2$, there is an $x' \in X'$ with $x'(x_1) \neq x'(x_2)$

Proof: $x := x_2 - x_1 \overset{(a)}{\Longrightarrow} \underbrace{x'(x)}_{= x'(x_2) - x'(x_1)} = \|x\|_X \neq 0 \Rightarrow x'(x_1) \neq x'(x_2)$

(c) For all $x \in X$: $\|x\|_X = \sup\{ |x'(x)| \mid x' \in X', \|x'\| = 1 \}$

Proof:

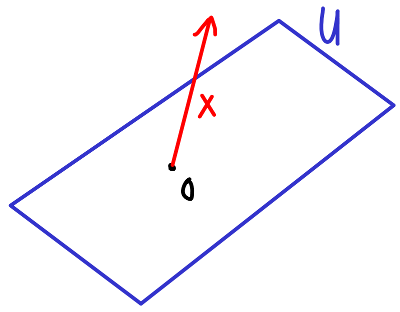
$$\|x'\|_{X'} \geq \frac{|x'(x)|}{\|x\|_X} \Rightarrow 1 = \sup_{\|x'\|=1} \|x'\|_{X'} \geq \sup_{\|x'\|=1} \frac{|x'(x)|}{\|x\|_X}$$

$$\Rightarrow \|x\|_X \geq \sup_{\|x'\|=1} |x'(x)|$$

Use (a): $\|x\|_X \leq \sup_{\|x'\|=1} |x'(x)|$

(d) Let $U \subseteq X$ be a closed subspace, $x \in X$ with $x \notin U$.

Then there exists $x' \in X'$ with $x'|_U = 0$ and $x'(x) \neq 0$.



Proof: $X/U := \{ [z] \mid z \in X \}$, $[z] := \{ z + u \mid u \in U \}$

$\|[z]\|_{X/U} := \inf_{u \in U} \|z + u\|_X$ $\rightsquigarrow$ $(X/U, \|\cdot\|_{X/U})$ normed space

$\overset{(a)}{\Longrightarrow}$ There is a $y' \in (X/U)'$ with $y'([x]) \neq 0$.

Define $x' \in X'$ by $x'(z) := y'([z])$ for $z \in X$.