Functional analysis — part 27

Bounded inverse theorem: X, Y Banach spaces, $T \in \mathcal{B}(X, Y)$.

Then: T bijective $\Rightarrow$ $T^{-1} \in \mathcal{B}(Y, X)$ (It's <u>continuous</u>)

<u>Proof:</u> T bijective $\underset{\text{open mapping theorem}}{\Longrightarrow}$ T open map $\Rightarrow$ T^{-1} continuous $\square$

<u>Counterexample:</u> $X = \left(C([0,1]), \|\cdot\|_{\infty} \right)$, $Y = \left(\{ f \in C^1([0,1]) \mid f(0) = 0 \}, \|\cdot\|_{\infty} \right)$ → not complete
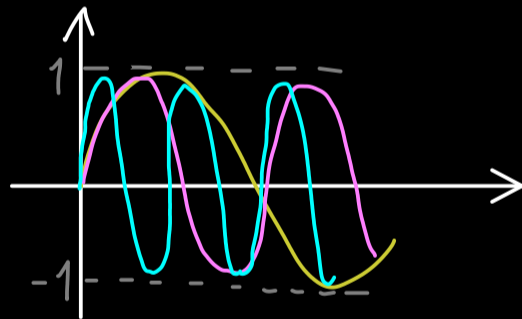
$$(Tf)(t) = \int_0^t f(s)\, ds \quad \text{linear and bounded } \text{and bijective}$$

$$\|Tf\|_{\infty} = \sup_{t \in [0,1]} \left| \int_0^t f(s)\, ds \right| \leq \|f\|_{\infty} \quad \Rightarrow \quad \|T\|_{X \to Y} \leq 1$$

Take $f_k(t) = \sin(kt)$

$$(Tf_k)(t) = \underbrace{\frac{1}{k}\left(1 - \cos(kt) \right)}_{g_k(t)}$$

$$T^{-1} g_k = f_k \quad \Rightarrow \quad \|T^{-1}\|_{Y \to X} \geq \frac{\|T^{-1} g_k\|_{\infty}}{\|g_k\|_{\infty}} = \frac{\overbrace{\|f_k\|_{\infty}}^{\leq 1}}{\underbrace{\|g_k\|_{\infty}}_{\leq \frac{2}{k}}} \geq \frac{k}{2} \xrightarrow{k \to \infty} \infty$$

$$\Rightarrow \quad T^{-1} \text{ \underline{not} continuous}$$