Functional analysis – part 28

Spectrum for bounded linear operators

Recall: $A \in \mathbb{C}^{n \times n}$ matrix with n rows and n columns.

$\lambda \in \mathbb{C}$ is called an eigenvalue of A if:

$$\exists x \in \mathbb{C}^n \setminus \{0\} : \quad Ax = \lambda x$$

$$\Leftrightarrow \exists x \in \mathbb{C}^n \setminus \{0\} : \quad (A - \lambda I) x = 0$$

$$\Leftrightarrow \quad \mathrm{Ker}(A - \lambda I) \neq \{0\}$$

$\Leftrightarrow$ map $x \mapsto (A - \lambda I)x$ not injective

Rank-nullity theorem: For any matrix $M \in \mathbb{C}^{m \times n}$:

$$\dim(\mathrm{Ran}(M)) + \dim(\mathrm{Ker}(M)) = n$$

Now: Let X be a complex Banach space and $T: X \to X$ be a bounded linear operator.

Definition: The spectrum of T is defined by: $\sigma(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not bijective} \}$

The resolvent set of T is defined by: $\rho(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ bijective and } (T - \lambda I)^{-1} \text{ bounded} \}$

bounded inverse theorem

$$\Longrightarrow \quad \sigma(T) = \mathbb{C} \setminus \rho(T)$$

We have the disjoint union: $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$

point spectrum: $\sigma_p(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not injective} \}$

continuous spectrum: $\sigma_c(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\mathrm{Ran}(T - \lambda I)} = X \}$

residual spectrum: $\sigma_r(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\mathrm{Ran}(T - \lambda I)} \neq X \}$