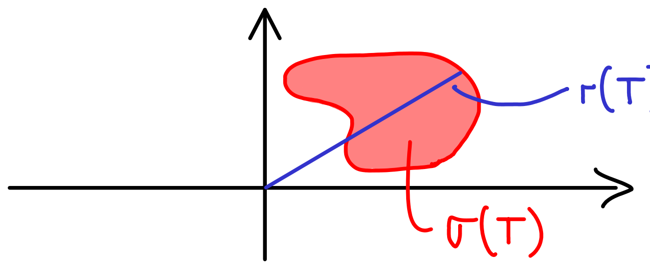
Functional analysis – part 31

Spectral radius: X complex Banach space $T: X \to X$ bounded linear operator



$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda|$$

Theorem: X complex Banach space, $T: X \to X$ bounded linear operator.

Then: (a) $\sigma(T) \subseteq \mathbb{C}$ is compact

(b) $X \neq \{0\} \Rightarrow \sigma(T) \neq \emptyset$

(c) $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{k \to \infty} \|T^k\|^{\frac{1}{k}} = \inf_{k \in \mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$

Proof: For $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$: $\left(I - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$

$$\Rightarrow \boxed{(T - \lambda)^{-1} = -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \qquad (*)}$$

$$\Rightarrow \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\| \Rightarrow \sigma(T) \text{ is bounded}$$

For (b): Assume $\sigma(T) = \emptyset \Rightarrow \rho(T) = \mathbb{C}$

Reminder: The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$$\lambda \longmapsto (T-\lambda)^{-1} \quad \text{is analytic.}$$

Take any $\ell \in \mathcal{B}(X)'$: $f_\ell : \mathbb{C} \to \mathbb{C}$

$$\lambda \mapsto \ell\left((T-\lambda)^{-1}\right)$$

analytic function (holomorphic function)

We get that f_ℓ is a bounded entire function.

$$\boxed{(T-\lambda)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{T}{\lambda}\right)^k \quad (*)}$$

For $|\lambda| \geq 2\cdot\|T\|$:

$$|f_\ell(\lambda)| \leq \|\ell\|\cdot\|(T-\lambda)^{-1}\| \leq \|\ell\| \underbrace{\frac{1}{|\lambda|}}_{\leq \frac{1}{2\|T\|}} \sum_{k=0}^{\infty} \underbrace{\left\|\frac{T}{\lambda}\right\|^k}_{\leq \frac{1}{2}}$$

$$\leq \frac{\|\ell\|}{\|T\|}$$

$$\overset{\text{Liouville's theorem}}{\Longrightarrow} \quad f_\ell \text{ is constant}$$

$$f_\ell(0) = \ell(T^{-1})$$

$$\|$$

$$f_\ell(\lambda) = \ell\left((T-\lambda)^{-1}\right) = \ell\left(\sum_{k=0}^{\infty}(T)^{-(k+1)}\cdot(\lambda)^k\right)$$

$$= \sum_{k=0}^{\infty}\ell\left(T^{-(k+1)}\right)\cdot\lambda^k$$

$$\Rightarrow \ell(T^{-2}) = 0 \quad \text{for all } \ell \in \mathcal{B}(X)'$$

$$\underset{\text{(part 25)}}{\overset{\text{Hahn–Banach theorem}}{\Longrightarrow}} \quad T^{-2} = 0 \quad \Rightarrow \quad X = \{0\}$$