Fundamental Theorem of Calculus

Theorem: If $f:[a,b] \longrightarrow \mathbb{R}$ is continuously differentiable,

then: $$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Riemann integral

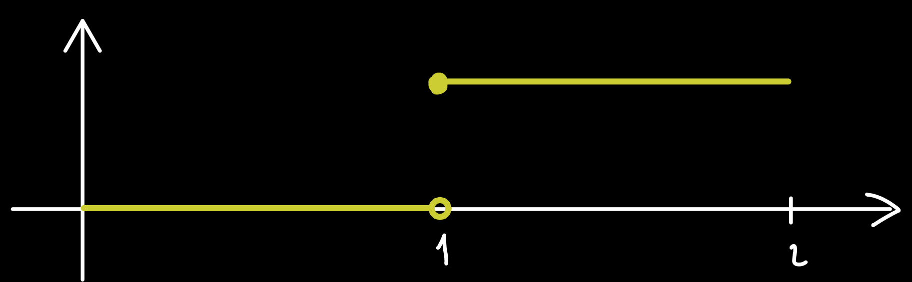
// Lebesgue integral

Remark: $\int_a^b f'(x)\, dx$ can exist even if $f'(x)$ is not defined for every $x \in [a,b]$.

First requirement: $f'(x)$ exists almost everywhere!

$\hookrightarrow$ $\{x \in [a,b] \mid f'(x) \text{ does not exist}\}$ has Lebesgue measure 0

Example:

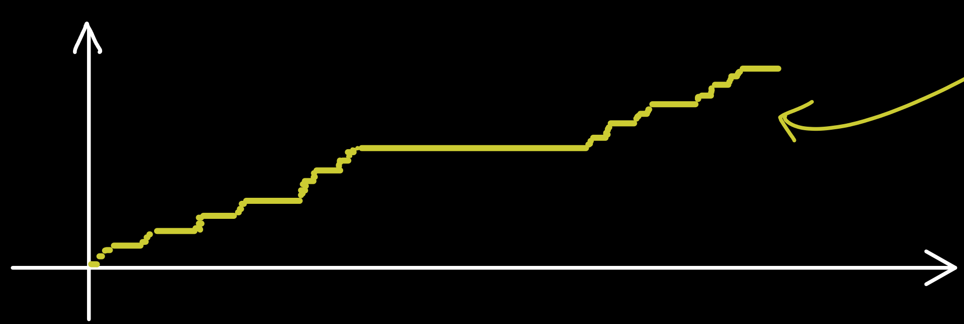


$$f(x) = \begin{cases} 0 \;, & x \in [0,1) \\ 1 \;, & x \in [1,2] \end{cases}$$

$f'(x) = 0$ for $x \neq 1$

$$\int_a^b f'(x)\, dx = 0 \neq 1 = f(b) - f(a)$$

Example:



Cantor function:

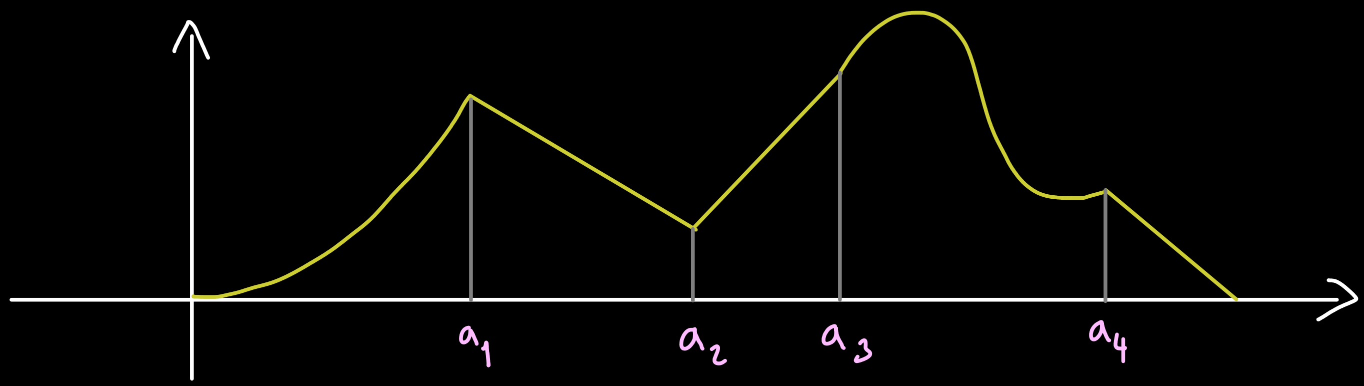
- continuous function
- $f'(x) = 0$ almost everywhere

Extension of the Fundamental Theorem of Calculus:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous and piecewise continuously differentiable,

which means there are $a =: a_0 < a_1 < a_2 < a_3 < \cdots < a_m < a_{m+1} := b$

such that $f|_{[a_j,\, a_{j+1}]}$ is continuously differentiable for each $j \in \{0, 1, \dots, m\}$.



Then: $$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Proof: $$\int_a^b f'(x)\, dx = \sum_{j=0}^{m} \int_{a_j}^{a_{j+1}} f'(x)\, dx = \sum_{j=0}^{m} \big(f(a_{j+1}) - f(a_j) \big)$$

original fundamental theorem of calculus

$$= f(b) - f(a) \qquad \square$$

Maximal extension of the Fundamental Theorem of Calculus:

For $f : [a,b] \longrightarrow \mathbb{R}$, we have the equivalence:

$$\int_a^c f'(x)\, dx = f(c) - f(a) \quad \text{for every } c \in [a,b]$$

$$\Longleftrightarrow$$ f is an absolutely continuous function