

Fundamental Theorem of Calculus

Theorem: If $f: [a, b] \rightarrow \mathbb{R}$ is continuously differentiable,

$$\text{then: } \int_a^b f'(x) dx = f(b) - f(a)$$

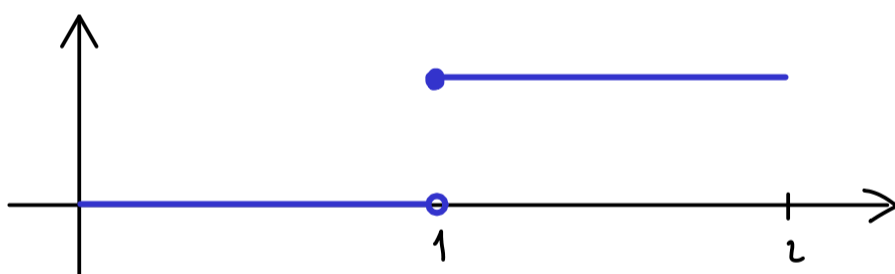
$\int_a^b f'(x) dx$ \equiv Riemann integral \equiv Lebesgue integral

Remark: $\int_a^b f'(x) dx$ can exist even if $f'(x)$ is not defined for every $x \in [a, b]$.

First requirement: $f'(x)$ exists almost everywhere!

$\hookrightarrow \{x \in [a, b] \mid f'(x) \text{ does not exist}\}$
has Lebesgue measure 0

Example:

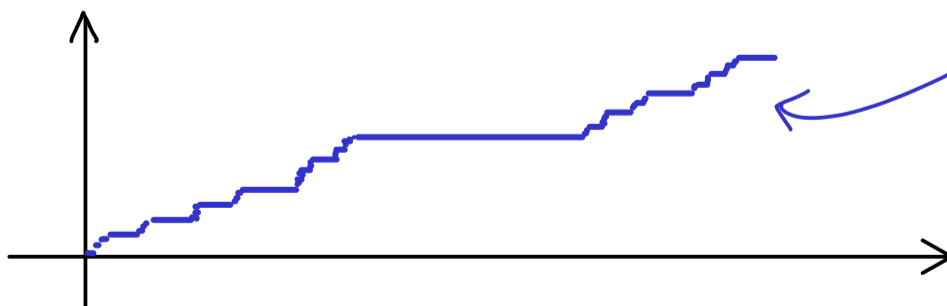


$$f(x) = \begin{cases} 0 & , x \in [0, 1) \\ 1 & , x \in [1, 2] \end{cases}$$

$$f'(x) = 0 \text{ for } x \neq 1$$

$$\int_a^b f'(x) dx = 0 \neq 1 = f(b) - f(a)$$

Example:

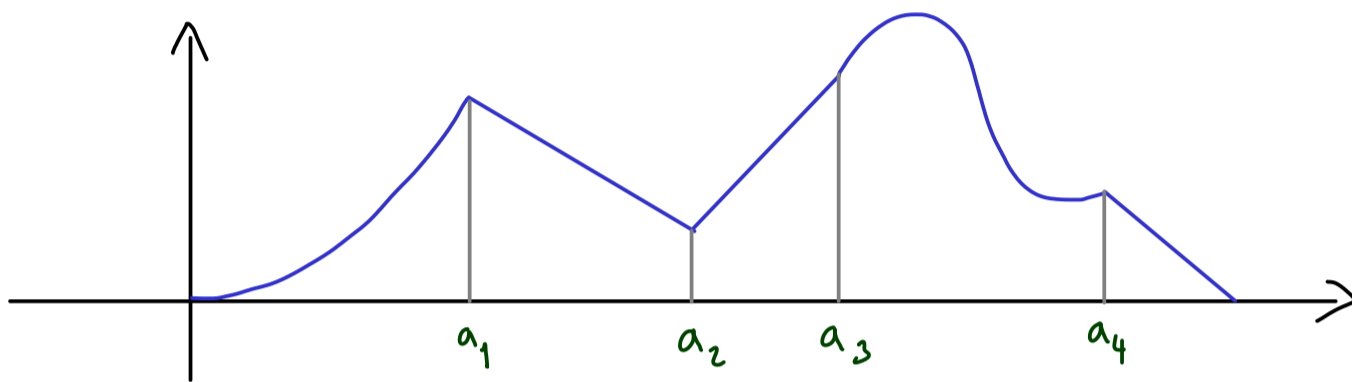


Cantor function:

- continuous function
- $f'(x) = 0$ almost everywhere

Extension of the Fundamental Theorem of Calculus:

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and piecewise continuously differentiable, which means there are $a =: a_0 < a_1 < a_2 < a_3 < \dots < a_m < a_{m+1} := b$ such that $f|_{[a_j, a_{j+1}]}$ is continuously differentiable for each $j \in \{0, 1, \dots, m\}$.



Then:
$$\int_a^b f'(x) dx = f(b) - f(a)$$

Proof:
$$\int_a^b f'(x) dx = \sum_{j=0}^m \int_{a_j}^{a_{j+1}} f'(x) dx \stackrel{\substack{\text{original fundamental} \\ \text{theorem of calculus}}}{=} \sum_{j=0}^m (f(a_{j+1}) - f(a_j)) = f(b) - f(a) \quad \square$$

Maximal extension of the Fundamental Theorem of Calculus:

For $f: [a, b] \rightarrow \mathbb{R}$, we have the equivalence:

$$\int_a^c f'(x) dx = f(c) - f(a) \quad \text{for every } c \in [a, b]$$

$\Leftrightarrow f$ is an absolutely continuous function