The Bright Side of Mathematics

Manifolds – Part 28

Wedge product: $\wedge$ multiplication defined for $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$

$$\wedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^s(V) \longrightarrow \mathrm{Alt}^{k+s}(V)$$
$$(\alpha, \beta) \longmapsto \alpha \wedge \beta$$

$(k+s)$-linear

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) :\neq \alpha(v_1, \dots, v_k) \cdot \beta(v_{k+1}, \dots, v_{k+s})$$

not a possible definition!
(not alternating)

Definition: For $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$, we define $\alpha \wedge \beta \in \mathrm{Alt}^{k+s}(V)$ by:

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) := \frac{1}{k! \cdot s!} \sum_{\sigma \in S_{k+s}} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+s)})$$

Examples: (a) $\alpha, \beta \in \mathrm{Alt}^1(V) = V^*$:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

(b) $\alpha, \beta \in \mathrm{Alt}^1(\mathbb{R}^3)$, $\alpha\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1$, $\beta\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_2 = \underbrace{(0, 1, 0)}_{\text{identified with } \beta} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$(\alpha \wedge \beta)\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 y_2 - y_1 x_2 = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{identified with } \alpha \wedge \beta} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

Properties: (a) $\alpha \wedge \beta = (-1)^{k \cdot s} \beta \wedge \alpha$ (anticommutative)

(b) $(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta$
$(\lambda \alpha) \wedge \beta = \lambda (\alpha \wedge \beta)$ (bilinear)

(c) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (associative)

(d) For a linear map $f : W \to V$ and $\alpha \in \mathrm{Alt}^k(V)$ define:

pullback $(f^* \alpha)(w_1, \dots, w_k) := \alpha(f(w_1), \dots, f(w_k))$

("natural")

$$f^*(\alpha \wedge \beta) = f^* \alpha \wedge f^* \beta$$