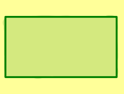
Manifolds – Part 30

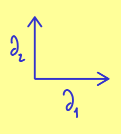
differential form on a manifold: $\omega \in \Omega^k(M)$ ← k-form on M + differentiable

$$\omega(p) = \sum_{\mu_1 < \cdots < \mu_k} \omega_{\mu_1, \mu_2, \ldots, \mu_k}(p) \cdot dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \cdots \wedge dx_p^{\mu_k}$$

Examples: (a) $M = \mathbb{R}^2$



$T_p M =$



$\partial_k = e_k$

$$dx_p^j(\partial_k) = \delta^j{}_k$$

identify: $\partial_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $dx_p^1 = (1, 0)$

$\partial_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $dx_p^2 = (0, 1)$

$$(dx_p^1 \wedge dx_p^2)(a_1, a_2) = \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma)\, dx_p^1(a_{\sigma(1)})\, dx_p^2(a_{\sigma(2)})$$

$a_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}$, $a_2 = \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$

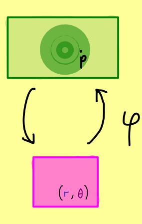
$$= \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} = \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

(b) Each $\omega \in \Omega^n(\mathbb{R}^n)$ can be written as:

$$\omega(p) = \omega_{1,2,\ldots,n}(p)\, dx_p^1 \wedge dx_p^2 \wedge \cdots \wedge dx_p^n$$

$$= \omega_{1,2,\ldots,n}(p) \det\left(\vdots \ \vdots \cdots \vdots \right)$$

(c) $M = \mathbb{R}^2$



φ given by polar coordinates $\varphi(r, \theta) = \begin{pmatrix} r \cdot \cos(\theta) \\ r \cdot \sin(\theta) \end{pmatrix}$

$$\partial_j := \varphi_*(e_j) = J_\varphi(\tilde{p})(e_j)$$

$$\partial_1(r, \theta) = \frac{\partial \varphi}{\partial r}(r, \theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$$\partial_2(r, \theta) = \frac{\partial \varphi}{\partial \theta}(r, \theta) = \begin{pmatrix} -r \cdot \sin(\theta) \\ r \cdot \cos(\theta) \end{pmatrix}$$

corresponding 1-forms: $dr_p = (\cos(\theta), \sin(\theta)) = \frac{1}{\sqrt{x^2+y^2}}(x, y)$

for $p = (x, y)$ $\quad d\theta_p = \frac{1}{r}(-\sin(\theta), \cos(\theta)) = \frac{1}{x^2+y^2}(-y, x)$

2-form: $(dr_p \wedge d\theta_p)(e_1, e_2) = dr_p(e_1)\, d\theta_p(e_2) - dr_p(e_2)\, d\theta_p(e_1)$

$$= \frac{1}{r}(\cos(\theta))^2 - \frac{1}{r} \cdot (-1)\, (\sin(\theta))^2$$

$$= \frac{1}{r}$$

$$\Longrightarrow \quad r\, dr_p \wedge d\theta_p = \det\left(\vdots \ \vdots \right) = dx_p \wedge dy_p$$