BECOME A MEMBER

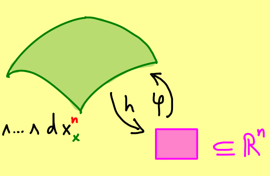
ON STEADY

The Bright Side of Mathematics

Manifolds – Part 36

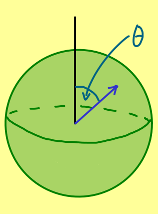
M orientable Riemannian manifold of dimension n.

↳ canonical volume form $\omega_M(x) = \sqrt{\det(G)}\, dx_x^1 \wedge \dots \wedge dx_x^n$



Examples: (a) $S^2 \subseteq \mathbb{R}^3$ has parameterization given by spherical coordinates:

$$\Phi(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$



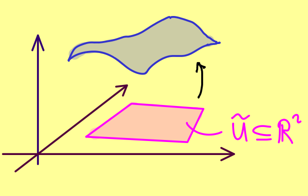
$$\Rightarrow \quad G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}$$

$$\Rightarrow \quad \omega_M(x) = \sin(\theta)\, d\theta \wedge d\varphi$$

(b) Graph surface: $f: \mathbb{R}^2 \to \mathbb{R}$ C^∞-function

$$M := \left\{ (x, f(x)) \mid x \in \mathbb{R}^2 \right\}$$

2-dim. submanifold in $\mathbb{R}^3$



Use parameterization: $\varphi: x \mapsto (x, f(x))$, $h: (x, f(x)) \mapsto x$

tangent vectors: $\partial_1^{(h)}(p) \overset{\text{identify}}{=} \frac{\partial \varphi}{\partial x_1}(x) = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x_1}(x) \end{pmatrix}$

$\partial_2^{(h)}(p) \overset{\text{identify}}{=} \frac{\partial \varphi}{\partial x_2}(x) = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x_2}(x) \end{pmatrix}$

$$g_{ij}^{(h)}(p) = \left\langle \frac{\partial \varphi}{\partial x_i}(x), \frac{\partial \varphi}{\partial x_j}(x) \right\rangle_{\text{standard}} = \begin{cases} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, & i \neq j \\ 1 + \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} & i = j \end{cases}$$

$$\Rightarrow \quad G = \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x_1}\right)^2 & \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} & 1 + \left(\frac{\partial f}{\partial x_2}\right)^2 \end{pmatrix}$$

$$\det(G) = 1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2$$

Canonical volume form: $\omega_M(p) = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2}\, dx_p^1 \wedge dx_p^2$

Interesting fact: $\left\| \partial_1^{(h)}(p) \times \partial_2^{(h)}(p) \right\|_{\text{standard}} = \left\| \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x_1}(x) \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x_2}(x) \end{pmatrix} \right\|_{\text{standard}}$

$$= \left\| \begin{pmatrix} -\frac{\partial f}{\partial x_1} \\ -\frac{\partial f}{\partial x_2} \\ 1 \end{pmatrix} \right\|_{\text{standard}} = \sqrt{\det(G)}$$