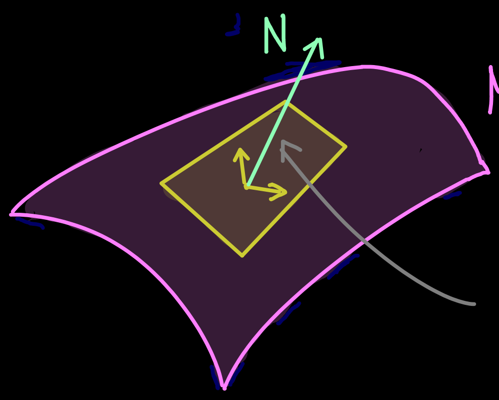
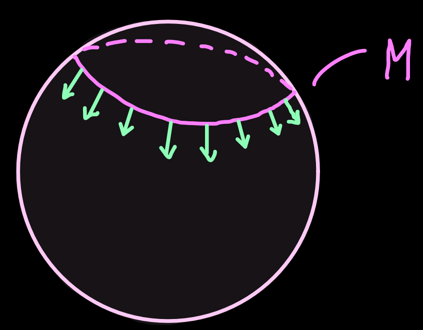
Manifolds — Part 37



$M \subseteq \mathbb{R}^3$ orientable Riemannian manifold of dimension 2

length of N $\iff$ canonical volume form

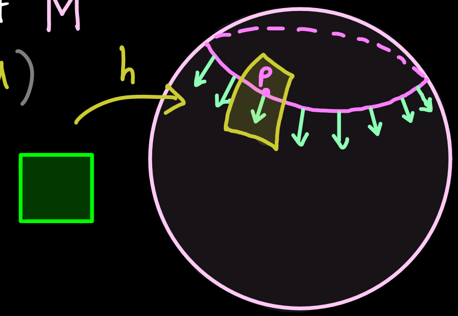
Definition: Let $\tilde{M}$ be a Riemannian manifold and $M \subseteq \tilde{M}$.



A map
$$N : M \longrightarrow T\tilde{M}$$
$$p \longmapsto N(p) \in T_p \tilde{M}$$
and $N(p) \in (T_p M)^{\perp} \setminus \{0\}$ (see $T_p M \subseteq T_p \tilde{M}$) (orthogonal w.r.t. g_p)

is called a normal vector field.

We call it continuous at p if for a chart (U, h) of $\tilde{M}$ $(p \in U)$ holds:



$$N(x) = \sum_i a_i(x) \cdot \partial_i^{(h)}(x)$$

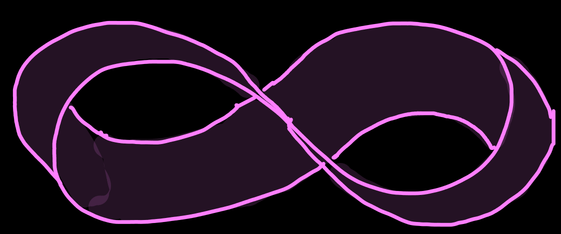
continuous functions $U \to \mathbb{R}$

We call it a continuous unit normal vector field if

- N is continuous at every $p \in M$
- $\|N(x)\| = \sqrt{g_x(N(x), N(x))} = 1$ for all $x \in M$.

Important fact: $M \subseteq \mathbb{R}^n$ $(n-1)$-dimensional submanifold:

(a) M is orientable $\iff$ M has a continuous unit normal vector field



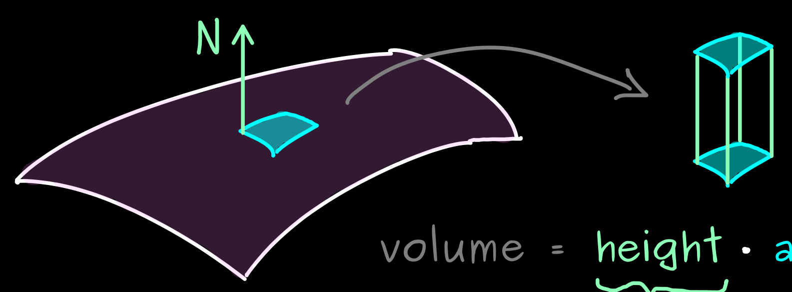
continuous normal vector field not possible

(b) If N is a continuous unit normal vector field, then:

canonical volume form $\longrightarrow$ $\omega_M = N \lrcorner \det$

means:

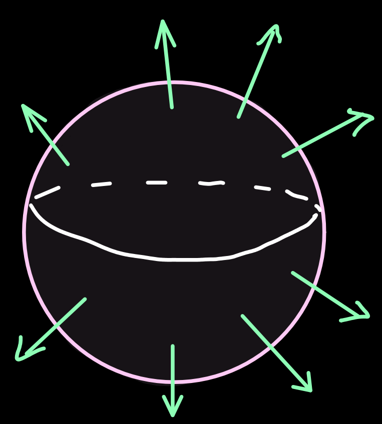
$$\omega_M(x)(v_1,\dots,v_{n-1}) = \det\big(N(x), v_1,\dots,v_{n-1}\big)$$



volume $=$ $\underbrace{\text{height}}_{=1} \cdot$ area

Example: $S^2 \subseteq \mathbb{R}^3$,

$N(x) = x$



parameterization:

$$\Phi(\theta,\varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

$$\sqrt{\det(G)} = \omega_M(x)\big(\partial_1^{(h)}(x), \partial_2^{(h)}(x)\big) = \det\big(N(x), \partial_1^{(h)}(x), \partial_2^{(h)}(x)\big)$$

$$= \det\begin{pmatrix} \sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & \sin(\theta)\cos(\varphi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{pmatrix}$$

$$= \sin(\theta)$$