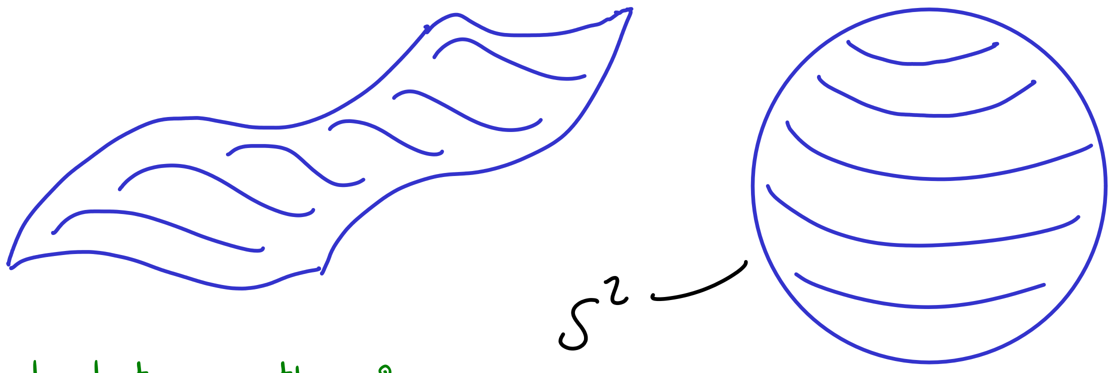
The Bright Side of Mathematics

The following pages cover the whole Manifolds course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Manifolds – Part 1

generalised surfaces?



How to calculate on them?

topology

differentiable manifolds

differential forms

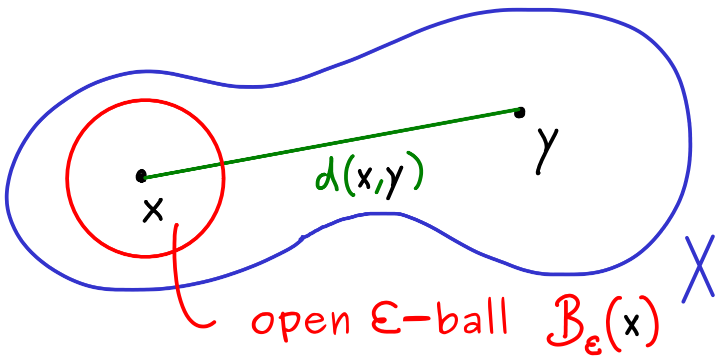
$d\omega$ $\int d\omega$

$\Longrightarrow$ (generalised) Stokes's Theorem

Metric space: (X, d)

set

distance function



⇝ define open sets $A \subseteq X$

Definition: Let X be a set, $\mathcal{P}(X)$ be the power set, and $\mathcal{T} \subseteq \mathcal{P}(X)$ be a collection of subsets.

If $\mathcal{T}$ satisfies:

(1) $\emptyset, X \in \mathcal{T}$

(2) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$

(3) $(A_i)_{i \in I}$ with $A_i \in \mathcal{T} \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$

then $\mathcal{T}$ is called a topology on X.

The elements of $\mathcal{T}$ are called open sets.

Examples:

(a) $\mathcal{T} = \{\emptyset, X\}$ is a topology on X (indiscrete topology)

(b) $\mathcal{T} = \mathcal{P}(X)$ is a topology on X (discrete topology)

Manifolds – Part 2

$\mathcal{T} \subseteq \mathcal{P}(X)$ topology on X:

(1) $\emptyset, X \in \mathcal{T}$

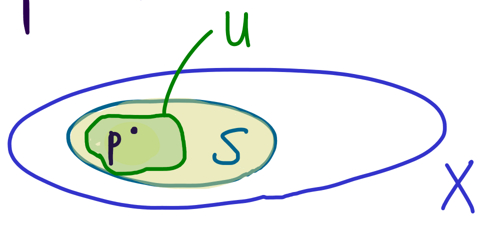
(2) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$

(3) $(A_i)_{i\in I}$ with $A_i \in \mathcal{T}$ $\Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$

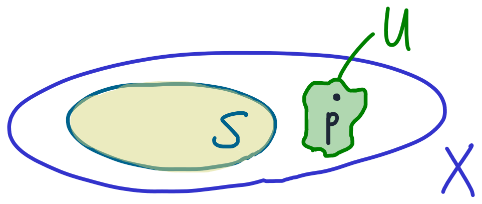
$(X, \mathcal{T})$ is called a topological space.

Important names: $(X, \mathcal{T})$ topological space, $S \subseteq X$, $p \in X$

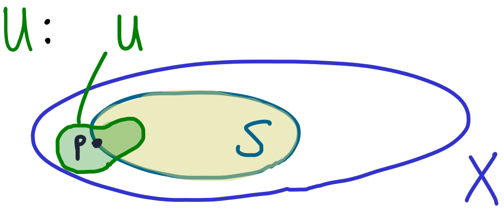
(a) p interior point of S $:\Longleftrightarrow$ There is an open set $U \in \mathcal{T}$: $p \in U$ and $U \subseteq S$



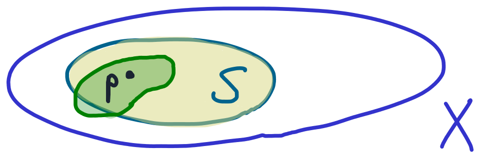
(b) p exterior point of S $:\Longleftrightarrow$ There is an open set $U \in \mathcal{T}$: $p \in U$ and $U \subseteq X \setminus S$



(c) p boundary point of S $:\Longleftrightarrow$ For all open sets $U \in \mathcal{T}$ with $p \in U$: $U \cap S \neq \emptyset$ and $U \cap (X \setminus S) \neq \emptyset$



(d) p accumulation point of S $:\Longleftrightarrow$ For all open sets $U \in \mathcal{T}$ with $p \in U$: $U \setminus \{p\} \cap S \neq \emptyset$



More names:

(a) $S^\circ := \{ p \in X \mid p \text{ interior point of } S \}$ interior of S

(b) $\mathrm{Ext}(S) := \{ p \in X \mid p \text{ exterior point of } S \}$ exterior of S

(c) $\partial S := \{ p \in X \mid p \text{ boundary point of } S \}$ boundary of S

(d) $S' := \{ p \in X \mid p \text{ accumulation point of } S \}$ derived set of S

(e) $\overline{S} := S \cup \partial S$ closure of S

Example: $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}$

$S = (0,1)$ ← not an open set!

← no interior points: there is no $\emptyset \neq U \in \mathcal{T}$ with $U \subseteq S$

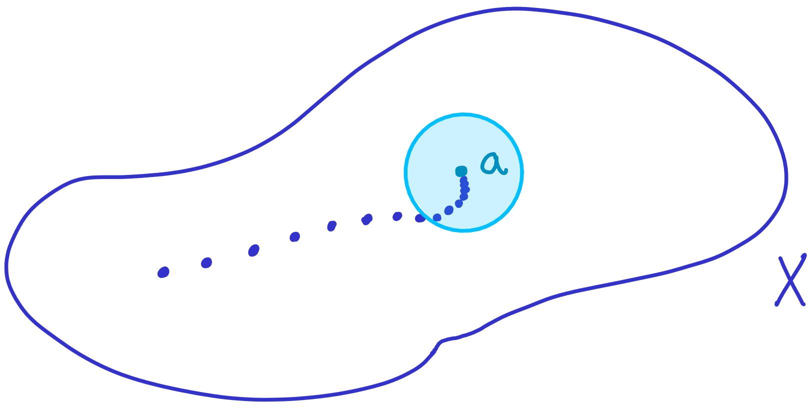
$$\Rightarrow S^{\circ} = \emptyset$$

$$X \setminus S = (-\infty, 0] \cup [1, \infty) \Rightarrow \operatorname{Ext}(S) = (1, \infty)$$

$$\Rightarrow \partial S = (-\infty, 1] \Rightarrow \overline{S} = (-\infty, 1]$$

Manifolds – Part 3

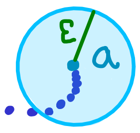
$(X, \mathcal{T})$ topological space



Convergence: $(a_n)_{n\in\mathbb{N}}$, $a_n \in X$

converges to $a \in X$

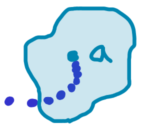
In a metric space:



The sequence members lie in each ε–ball around a, eventually.

For each ε–ball $B_\varepsilon(a)$, there is $N \in \mathbb{N}$ such that for all $n \geq N$: $a_n \in B_\varepsilon(a)$

In a topological space:



$\overbrace{\text{an open set } U \in \mathcal{T}}^{\text{open neighbourhood of } a}$ with $a \in U$

Definition: $(X, \mathcal{T})$ topological space, $(a_n)_{n\in\mathbb{N}}$ sequence in X.

$$a_n \xrightarrow{n \to \infty} a \;:\Longleftrightarrow$$

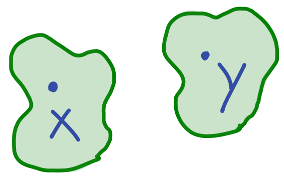
For each $U \in \mathcal{T}$ with $a \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$: $a_n \in U$

Example: $X = \mathbb{R}$, $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$$

- converges to 0 : each open neighbourhood of 0 looks like (b, ∞) for $b < 0$, so: $\frac{1}{n} \in (b, \infty)$
- converges to -1 : each open neighbourhood of -1 looks like (b, ∞) for $b < -1$, so: $\frac{1}{n} \in (b, \infty)$
- converges to -2

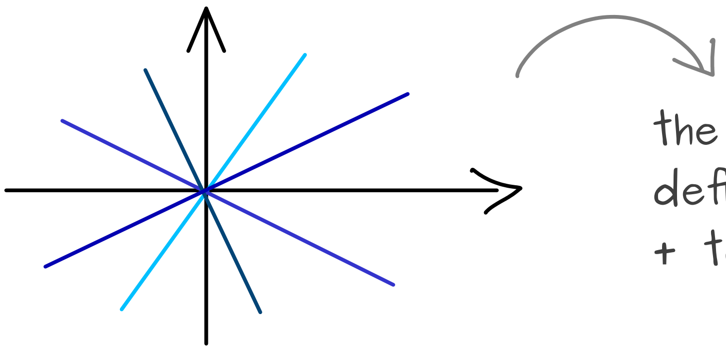
Definition: A topological space $(X, \mathcal{T})$ is called a Hausdorff space if for all $x, y \in X$ with $x \neq y$ there is an open neighbourhood of x: $U_x \in \mathcal{T}$ and there is an open neighbourhood of y: $U_y \in \mathcal{T}$ with:



$$U_x \cap U_y = \emptyset$$

Manifolds – Part 4

Projective space: $P^n(\mathbb{R})$ = set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$



the directions define a set + topology?

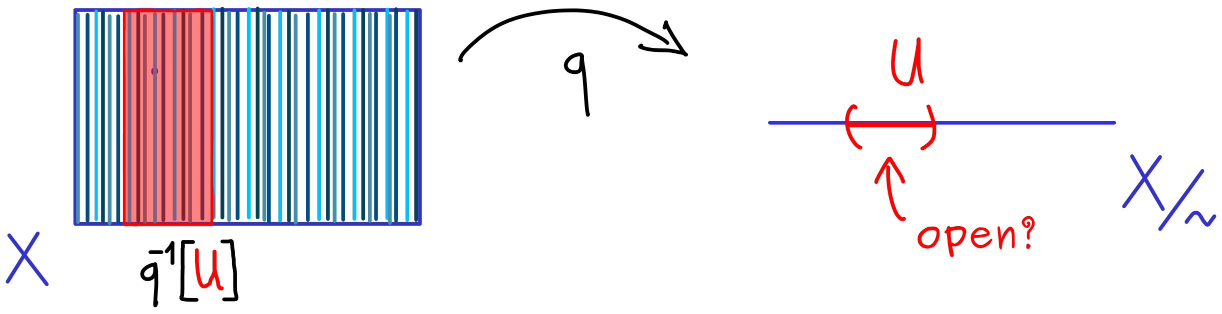
Quotient topology: $(X, \mathcal{T})$ topological space, $\sim$ equivalence relation on X

↳ reflexive $x \sim x$
symmetric $x \sim y \Rightarrow y \sim x$
transitive $x \sim y \wedge y \sim z \Rightarrow x \sim z$

equivalence class of x : $[x]_\sim := \{ y \in X \mid y \sim x \}$

$X/_\sim := \{ [x]_\sim \mid x \in X \}$ quotient set

$q : X \to X/_\sim \,,\quad x \mapsto [x]_\sim$ canonical projection



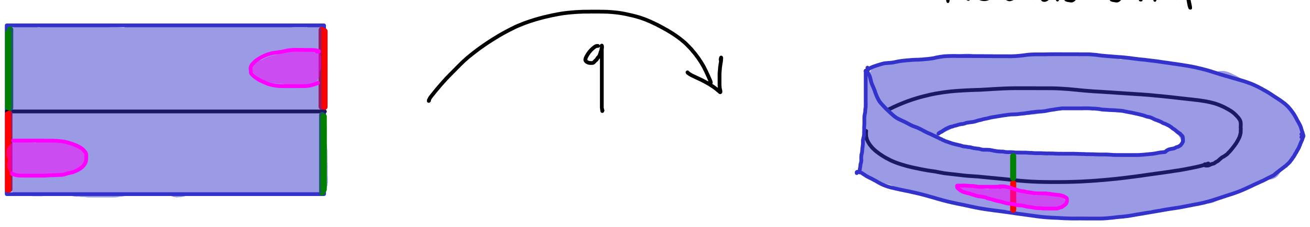
$$q^{-1}[U] \subseteq X \text{ open } \Longleftrightarrow: U \subseteq X/_\sim \text{ open}$$

$$q^{-1}[U] \in \mathcal{T} \Longleftrightarrow: U \in \hat{\mathcal{T}}$$

This defines a topology $\hat{\mathcal{T}}$ on $X/_\sim$, called the quotient topology.

Example:

$X = [0,1] \times (-1,1)$



Möbius strip

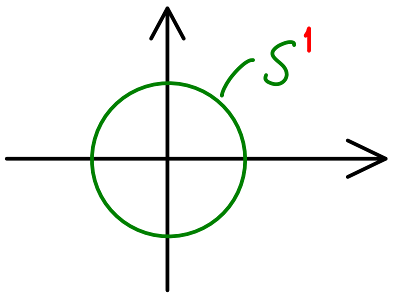
equivalence relation: $(0,s) \sim (1,-s)$

Manifolds – Part 5

$(X, \mathcal{T})$ topological space $\rightsquigarrow$ $(X/\sim, \hat{\mathcal{T}})$ quotient space

Projective space: $P^n(\mathbb{R}) =$ set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$

$S^n \subseteq \mathbb{R}^{n+1}$

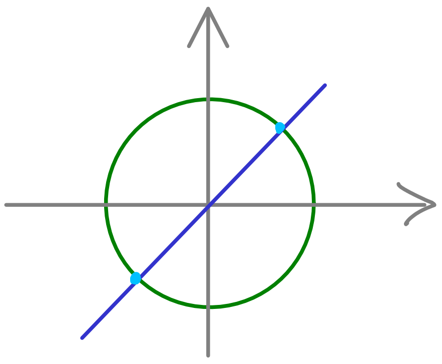


$$S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

Euclidean norm

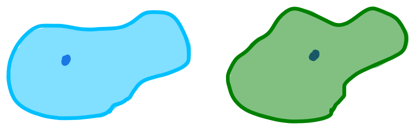
equivalence relation: $x \sim -x$

Let's define: $x \sim y :\Longleftrightarrow (x = y \text{ or } x = -y)$



$P^n(\mathbb{R}) := S^n/\sim$ with quotient topology

Is $P^n(\mathbb{R})$ a Hausdorff space?



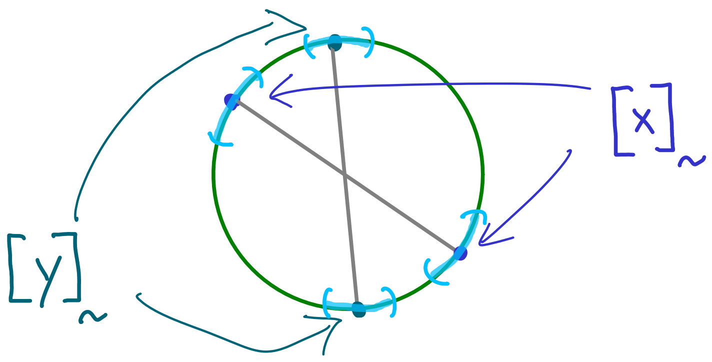
Take $[x]_\sim, [y]_\sim \in P^n(\mathbb{R})$ with $[x]_\sim \neq [y]_\sim \Rightarrow x \neq y$ and $x \neq -y$

Take open neighbourhoods

$U, V \subseteq S^n$ of x and y, respectively,

with $U \cap V = \emptyset$, $-U \cap V = \emptyset$

$-U \cap -V = \emptyset$, $U \cap -V = \emptyset$



Look at: $\hat{U} := q[U]$, $\quad q: S^n \to S^n/\sim$ canonical projection

$$q^{-1}[\hat{U}] = U \cup (-U) \underset{\nwarrow \text{ open}}{\in} \mathcal{T} \quad \Rightarrow \quad \hat{U} \underset{\nwarrow \text{ open}}{\in} \hat{\mathcal{T}}$$

(the same for $\hat{V} := q[V]$)

We find: $q^{-1}[\hat{U} \cap \hat{V}] = q^{-1}[\hat{U}] \cap q^{-1}[\hat{V}] = (U \cup (-U)) \cap (V \cup -V) = \emptyset$

$$\overset{q \text{ surjective}}{\Longrightarrow} \quad \hat{U} \cap \hat{V} = \emptyset$$

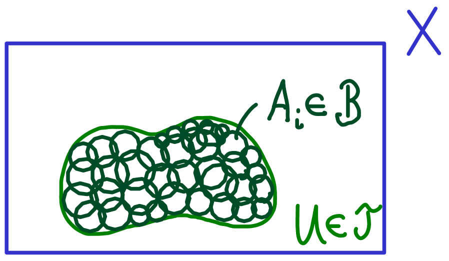
Manifolds – Part 6

$(X, \mathcal{T})$ topological space: generate the topology $\mathcal{T}$

Definition: Let $(X, \mathcal{T})$ be a topological space. A collection of open subsets $\mathcal{B} \subseteq \mathcal{T}$ is called a basis (base) of $\mathcal{T}$ if:

for all $U \in \mathcal{T}$ there is $(A_i)_{i \in I}$ with $A_i \in \mathcal{B}$

and $\bigcup_{i \in I} A_i = U$



Examples: (a) $\mathcal{B} = \mathcal{T}$ is always a basis.

(b) If $\mathcal{T}$ is discrete topology on X, then $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis of $\mathcal{T}$.

(c) Let $(X, \mathcal{T})$ be the topological space induced by a metric space (X, d)

$\mathcal{B} = \{B_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$ is a basis of $\mathcal{T}$.

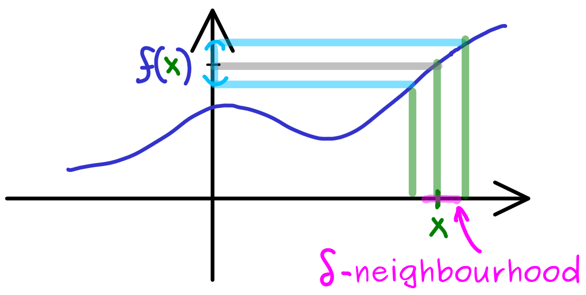
(d) $\mathbb{R}^n$ with standard topology (defined by Euclidean metric)

$\mathcal{B} = \{B_\varepsilon(x) \mid x \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}, \varepsilon > 0\}$ is a basis of $\mathcal{T}$.

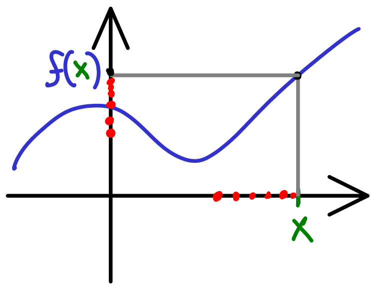
↗ only countably many elements

Definition: A topological space $(X, \mathcal{T})$ is called second-countable if there is a countable basis of $\mathcal{T}$.

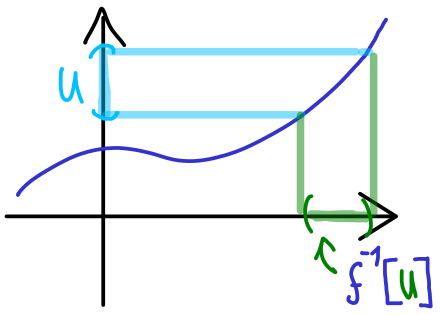
Manifolds - Part 7



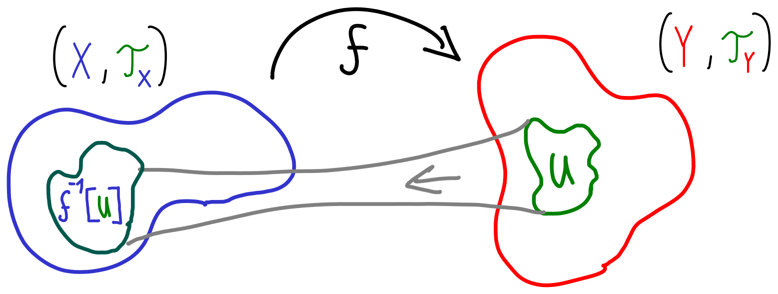
ε-δ-definition



sequence definition



general definition



Definition: $(X,\mathcal{T}_X), (Y,\mathcal{T}_Y)$ topological spaces.

$f: X \longrightarrow Y$ is called continuous if

$$U \in \mathcal{T}_Y \Rightarrow f^{-1}[U] \in \mathcal{T}_X .$$

homeomorphism = $f: X \longrightarrow Y$ bijective, continuous and $f^{-1}: Y \to X$ continuous

Examples: (a) $(Y,\mathcal{T}_Y)$ = indiscrete topological space $\Rightarrow$ $f: X \to Y$ continuous

(b) $(X,\mathcal{T}_X)$ = discrete topological space $\Rightarrow$ $f: X \to Y$ continuous

(c) $(X,\mathcal{T}_X)$ with equivalence relation $\sim$, $(X/\sim , \hat{\mathcal{T}})$ quotient space

$q: X \to X/\sim , \quad x \mapsto [x]_\sim$ canonical projection is continuous

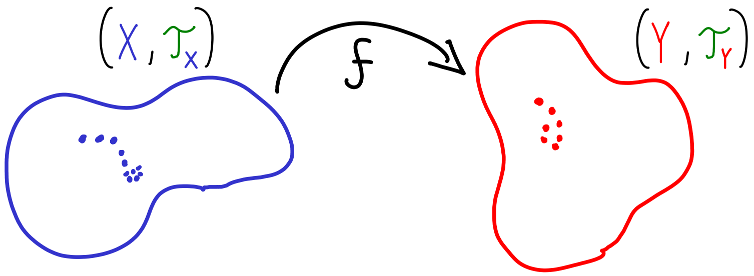
Definition: $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ topological spaces.

$f : X \longrightarrow Y$ is called sequentially continuous if for all $x \in X$:

$(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} x$

$$\Longrightarrow$$

$(f(x_n))_{n\in\mathbb{N}} \subseteq Y$ convergent with $f(x_n) \xrightarrow{n\to\infty} f(x)$



Fact:

$f : X \to Y$ continuous $\Longleftrightarrow$ $f : X \to Y$ sequentially continuous

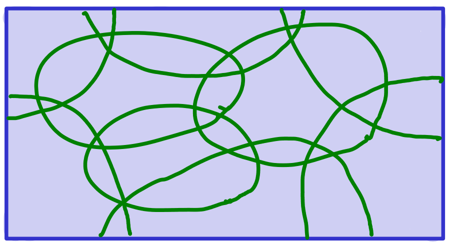
in metric spaces

second-countable spaces

Manifolds – Part 8

$[a,b] \subseteq \mathbb{R}$ compact (Bolzano–Weierstrass and Heine–Borel)

$(X, \mathcal{T})$

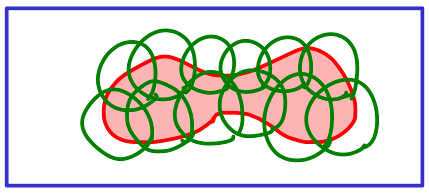


cover with open sets
$\rightsquigarrow$
do finitely many suffice?

Definition: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

A is called compact if

$$\bigcup_{i \in I} U_i \supseteq A \text{ with } U_i \in \mathcal{T} \Rightarrow \text{there is a finite } I_0 \subseteq I \text{ with: } \bigcup_{i \in I_0} U_i \supseteq A$$



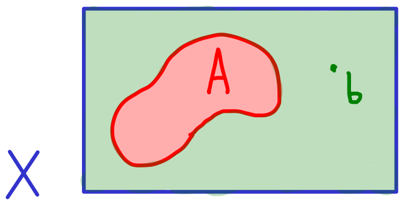
We know: $A \subseteq \mathbb{R}^n$ compact $\Longleftrightarrow$ A closed and bounded (Heine–Borel theorem)

$\uparrow$ with standard topology

Proposition: Let $(X, \mathcal{T})$ be a Hausdorff space. Then:

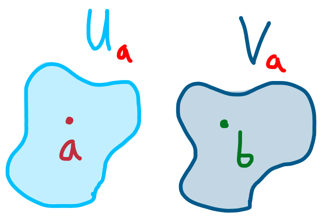
$$A \subseteq X \text{ compact} \Rightarrow A \text{ closed} \quad \begin{pmatrix} X \setminus A \text{ open} \\ X \setminus A \in \mathcal{T} \end{pmatrix}$$

Proof:

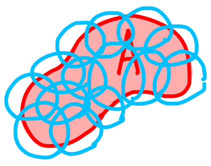


Assume A is compact.

Fix $b \in X \setminus A$.

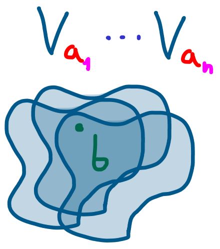


For any $a \in A$, there are $U_a, V_a \in \mathcal{T}$ with $a \in U_a$, $b \in V_a$ and $U_a \cap V_a = \emptyset$



$$A \subseteq \bigcup_{a \in A} U_a$$ (open cover)

$$\overset{A \text{ compact}}{\Longrightarrow} \quad A \subseteq \bigcup_{j=1}^{n} U_{a_j}$$ (finite subcover)



$$\Longrightarrow \quad V := \bigcap_{j=1}^{n} V_{a_j}$$ open neighbourhood of b

with $A \cap V \subseteq \bigcup_{j=1}^{n} U_{a_j} \cap \bigcap_{j=1}^{n} V_{a_j} = \emptyset$

$\Longrightarrow$ b is an interior point of $X \setminus A$ $\Longrightarrow$ A closed

Manifolds – Part 9

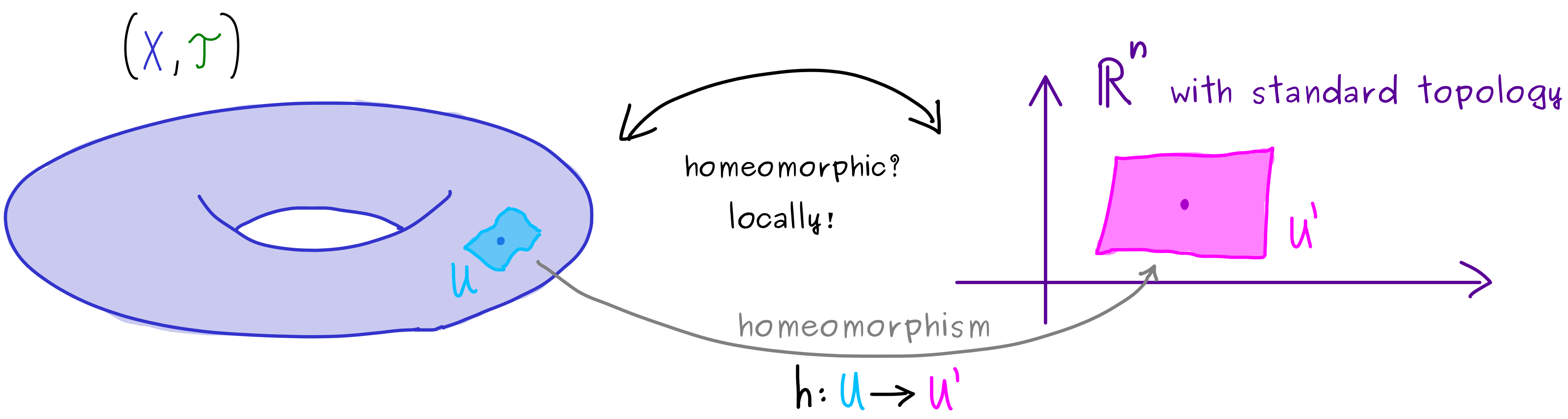
Definition: n –dimensional (topological) manifold:

topological space $(X, \mathcal{T})$ with:

(1) Hausdorff space

(2) second-countable

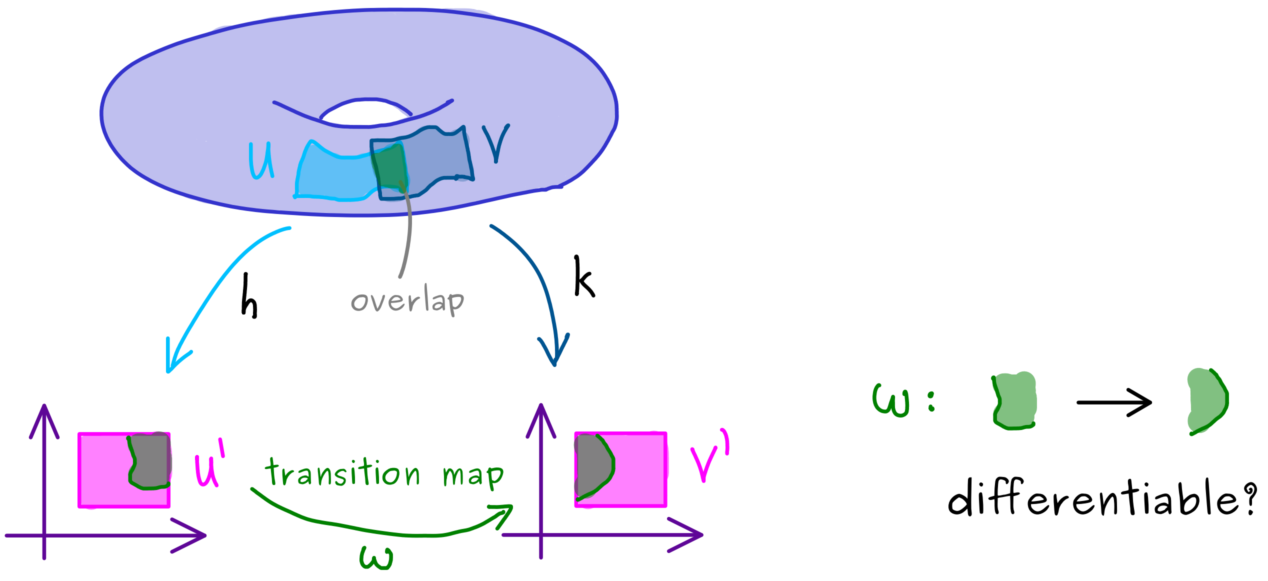
(3) locally Euclidean of dimension n



Definition: $(X, \mathcal{T})$ is called locally Euclidean of dimension n if:

For all $x \in X$ there is an open neighbourhood $U \in \mathcal{T}$ and a homeomorphism $h: U \to U'$ with $U' \subseteq \mathbb{R}^n$ open.

The map $h: U \to U'$ is called a chart of $(X, \mathcal{T})$.



Manifolds – Part 10

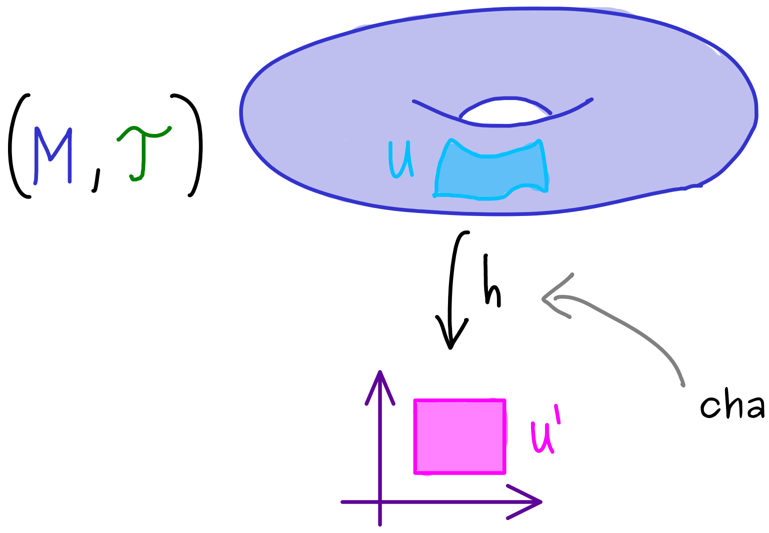


chart (U, h)

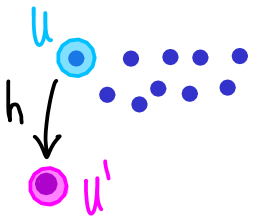
(1) Hausdorff space

(2) second-countable

(3) locally Euclidean of dimension n

Definition: A collection of charts $(U_i, h_i)_{i \in I}$ is called an atlas if: $\bigcup_{i \in I} U_i = M$

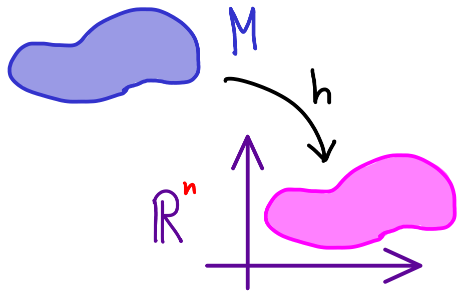
Example: (a) $(M, \mathcal{T})$ discrete topological space with countably many points



⇝ 0 - dimensional manifold

(b) $M \subseteq \mathbb{R}^n$ open subset , $(M, \mathcal{T})$

with standard topology

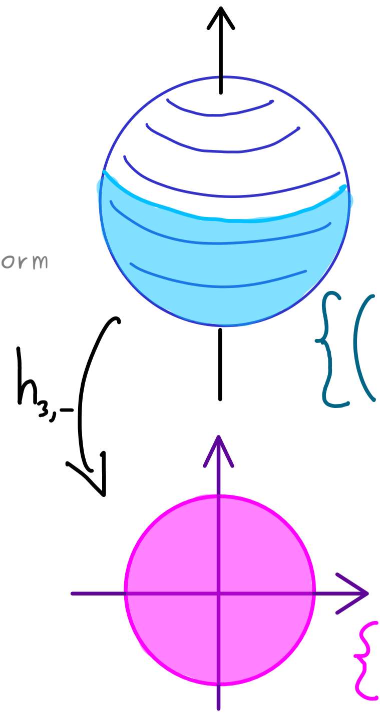


⇝ n - dimensional manifold

(c) $S^2 \subseteq \mathbb{R}^3$, $S^2 := \left\{ x \in \mathbb{R}^3 \;\middle|\; \|x\| = 1 \right\}$

Euclidean norm

2 - dimensional manifold



$U_{3,-} := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_3 < 0 \right\}$

$U'_{3,-} := \left\{ x' \in \mathbb{R}^2 \;\middle|\; \|x'\| < 1 \right\}$

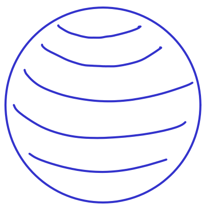
$$h_{3,-} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$h_{3,-}^{-1} : \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \longmapsto \begin{pmatrix} x'_1 \\ x'_2 \\ -\sqrt{1 - \|x'\|^2} \end{pmatrix}$$

$(U_{i,\pm}, h_{i,\pm})_{i \in \{1,2,3\}}$ is an atlas.

Manifolds – Part 11

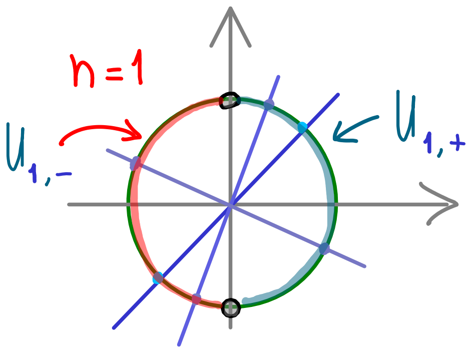
$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$



is an n-dimensional manifold with atlas $\Big(\overset{\{x \in \mathbb{R}^{n+1} \mid \pm x_i > 0\} \,=:}{U_{i,\pm}}, h_{i,\pm}\Big)_{i \in \{1,\ldots,n+1\}}$

Projective space: $P^n(\mathbb{R}) := S^n/\sim$ with quotient topology

equivalence relation: $x \sim y :\Leftrightarrow (x = y \text{ or } x = -y)$



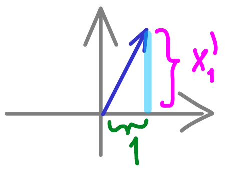
$$\begin{aligned} q: S^n &\to S^n/\sim \\ x &\mapsto [x]_\sim \end{aligned}$$ canonical projection

$V_i := \{[x]_\sim \in P^n(\mathbb{R}) \mid x_i \neq 0\}$, $q^{-1}[V_i] = U_{i,+} \cup U_{i,-}$

↳ open

for $n = 1$: $h_1 : V_1 \to V_1' \subseteq \mathbb{R}^1$, $h_1([x]_\sim) = \frac{x_2}{x_1}$ slope

with inverse $h_1^{-1}(x_1') = \left[\begin{pmatrix} 1 \\ x_1' \end{pmatrix} \cdot \frac{1}{\sqrt{1^2 + (x_1')^2}}\right]_\sim$



h_2 works similarly $\Longrightarrow$ 1-dimensional manifold

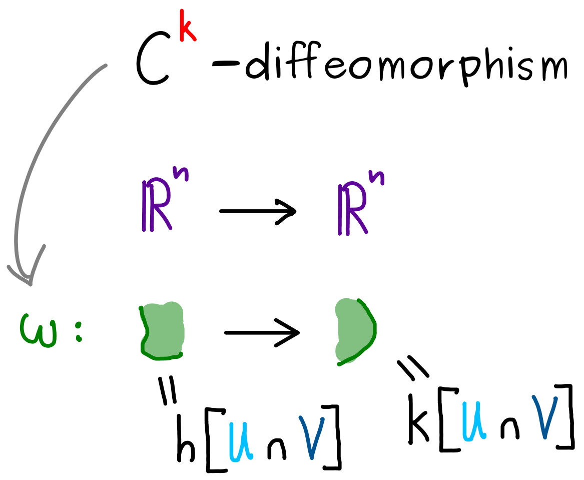
for $n \in \mathbb{N}$: $h_i : V_i \to V_i' \subseteq \mathbb{R}^n$

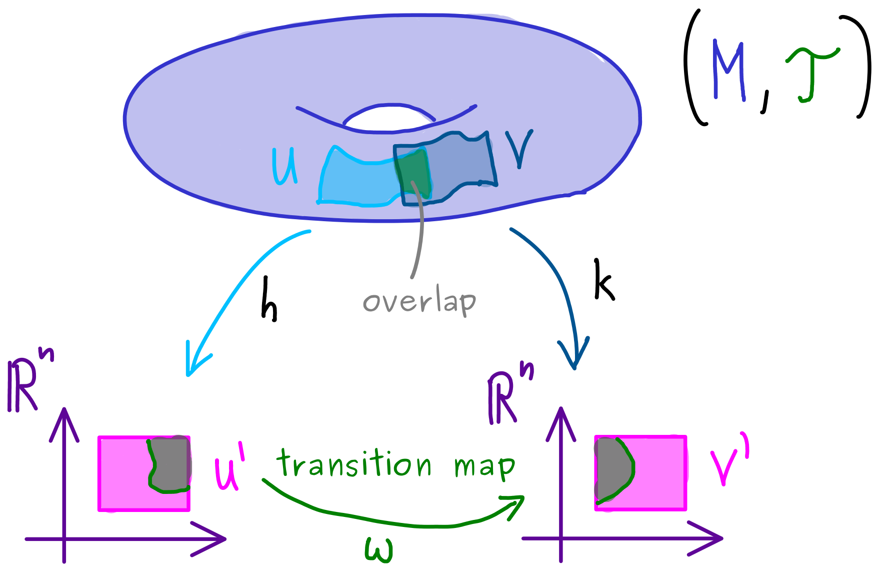
$$h_i([x]_\sim) = \begin{pmatrix} \frac{x_1}{x_i} \\ \vdots \\ \frac{x_{i-1}}{x_i} \\ \frac{x_{i+1}}{x_i} \\ \vdots \\ \frac{x_{n+1}}{x_i} \end{pmatrix}$$ homeomorphism

$\Longrightarrow$ n-dimensional manifold

Manifolds – Part 12

Smooth structures





C^k-diffeomorphism : $k \in \{0, 1, \ldots\}$ or $k = \infty$

- ω is k-times continuously differentiable (partial derivatives up to the k-th order exist and are continuous) $\Big\}\ \omega \in C^k(\cdot)$
- ω is bijective
- $\omega^{-1} \in C^k(\cdot\cdot)$

Definition:

- Two charts h, k are called C^k- smoothly compatible if the transition map is a C^k-diffeomorphism.
- An atlas $\{(U_i, h_i)_{i \in I}\}$ is called a C^k-atlas if any two charts are C^k- smoothly compatible.
- A maximal C^k-atlas $\mathcal{A}$ is: (1) $\mathcal{A}$ is a C^k-atlas
  (2) For any other C^k-atlas $\mathcal{B}$, we have $\mathcal{B} \not\supsetneq \mathcal{A}$.

Definition: n-dimensional C^k-smooth manifold:

- n-dimensional (topological) manifold
- maximal C^k-atlas (C^k-smooth structure)

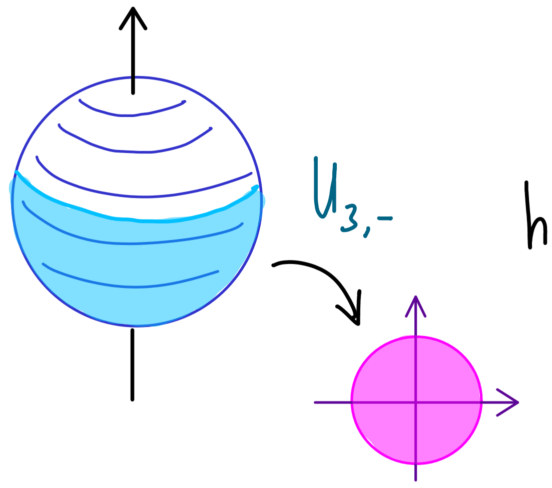
Manifolds – Part 13

Examples for smooth manifolds:

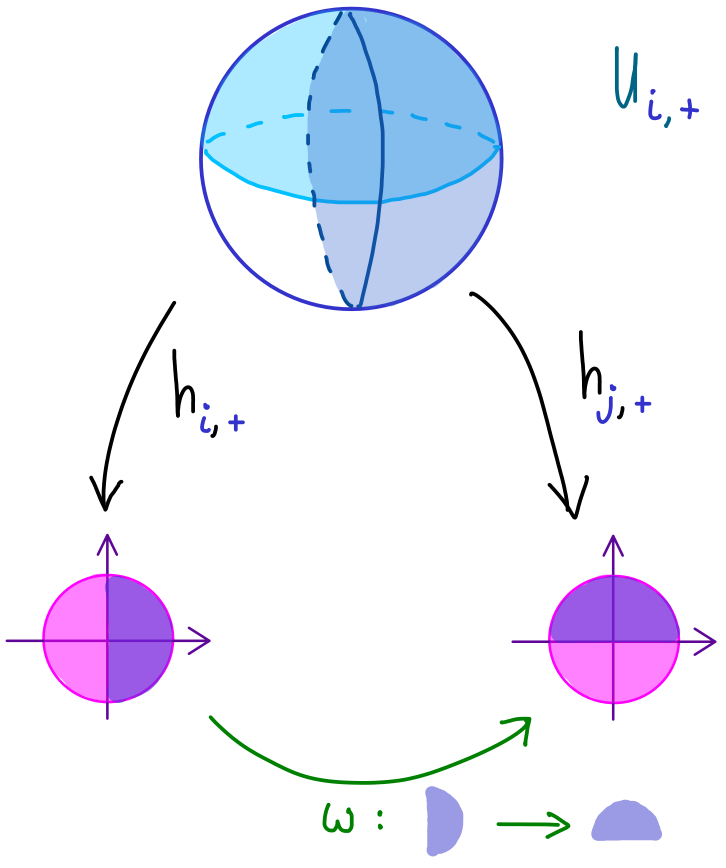
(a) $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth manifold.

We show that $\left(U_{i,\pm}, h_{i,\pm}\right)_{i \in \{1,\dots,n+1\}}$ is C^∞-atlas:

$$\underset{\{x \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}}{\underset{\shortparallel}{U_{i,\pm}}}$$



$$h_{i,\pm} : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_{i+1} \\ \vdots \\ x_{n+1} \end{pmatrix}$$



$U_{i,+} \cap U_{j,+}$

$\left(h_{j,+} \circ h_{i,+}^{-1}\right)$

For $n = 2$, $i = 3$, $j = 1$

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \xrightarrow{h_{i,+}^{-1}} \begin{pmatrix} x_1' \\ x_2' \\ \sqrt{1-\|x'\|^2} \end{pmatrix} \xmapsto{h_{j,+}} \begin{pmatrix} x_2' \\ \sqrt{1-\|x'\|^2} \end{pmatrix}$$

C^∞-diffeomorphism

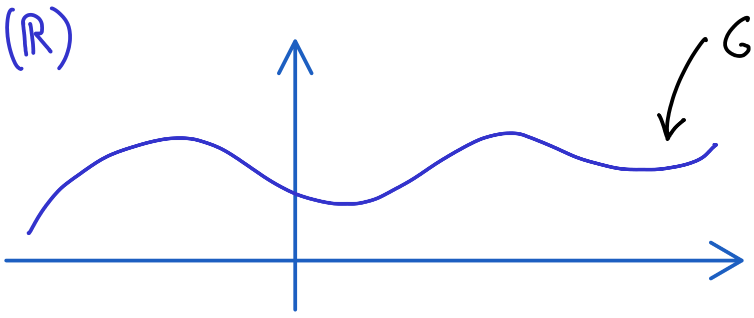
⇝ extend to a maximal C^∞-atlas ⇝ C^∞-smooth manifold

(b) $\mathbb{R}^n$ is a smooth manifold

$\hookrightarrow$ atlas given by one chart $(\mathbb{R}^n, id)$ $\rightsquigarrow$ extend to a maximal C^∞-atlas

(standard smooth structure for $\mathbb{R}^n$)

(c) Consider $f \in C^1(\mathbb{R})$



$G_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$

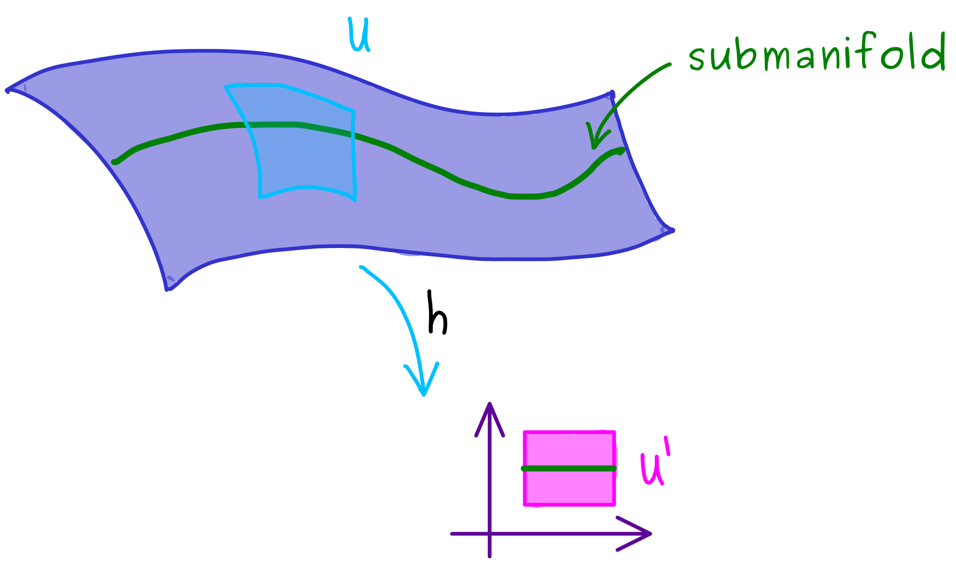
$\subseteq \mathbb{R} \times \mathbb{R}$

G_f is a 1-dimensional manifold with one chart: $h: G_f \to \mathbb{R}$

$(x, f(x)) \mapsto x$

$\rightsquigarrow$ extend to a smooth structure

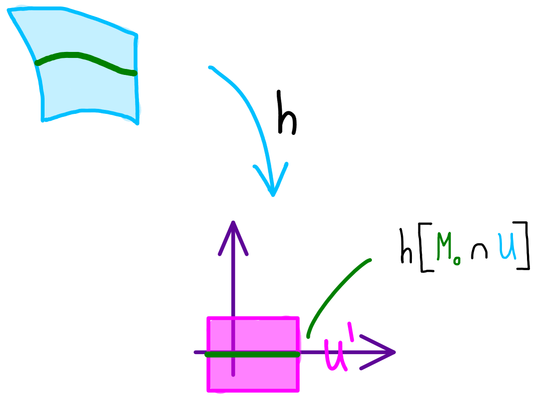
Manifolds - Part 14



Definition: Let M be an n-dimensional (smooth) manifold.
$M_0 \subseteq M$ is called a k-dimensional submanifold of M if

for all $p \in M_0$ there is a chart (U, h) of M with

$$h[M_0 \cap U] = (\mathbb{R}^k \times \underbrace{0}_{n-k \text{ zeros}}) \cap U'$$



(U, h) is called a submanifold chart for M_0.

Note: M_0 is also a manifold:

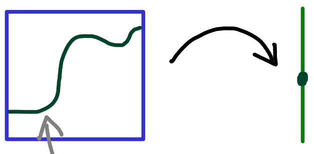
(U, h) submanifold chart $\rightsquigarrow$ $(\tilde{U}, \tilde{h})$ chart, $\tilde{U} := U \cap M_0$

$$\tilde{h} \text{ given by } p \mapsto h(p) = \begin{pmatrix} \circledast \\ \vdots \\ \circledast \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} \circledast \\ \vdots \\ \circledast \end{pmatrix} \in \mathbb{R}^k$$

Manifolds – Part 15

Regular value theorem in $\mathbb{R}^n$ = preimage theorem = submersion theorem

$f: \mathbb{R}^n \to \mathbb{R}^m$ smooth



preimage = smooth submanifold?

Definition: $f: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ open, C^1-function.

(1) $x \in U$ is called a critical point of f if

df_x is not surjective (or $J_f(x)$ has rank less than m)

(2) $c \in f[U]$ is called a regular value of f if

$f^{-1}[\{c\}]$ does not contain any critical points.

Theorem: $f: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ open, C^∞-function. $(n \geq m)$

If c is a regular value of f, then

$f^{-1}[\{c\}]$ is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$.

Proof: Use implicite function theorem.

Example: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x_1, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$

$$J_f(x_1, \dots, x_n) = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \end{pmatrix}$$

$\Rightarrow$ $x = 0$ is the only critical point.

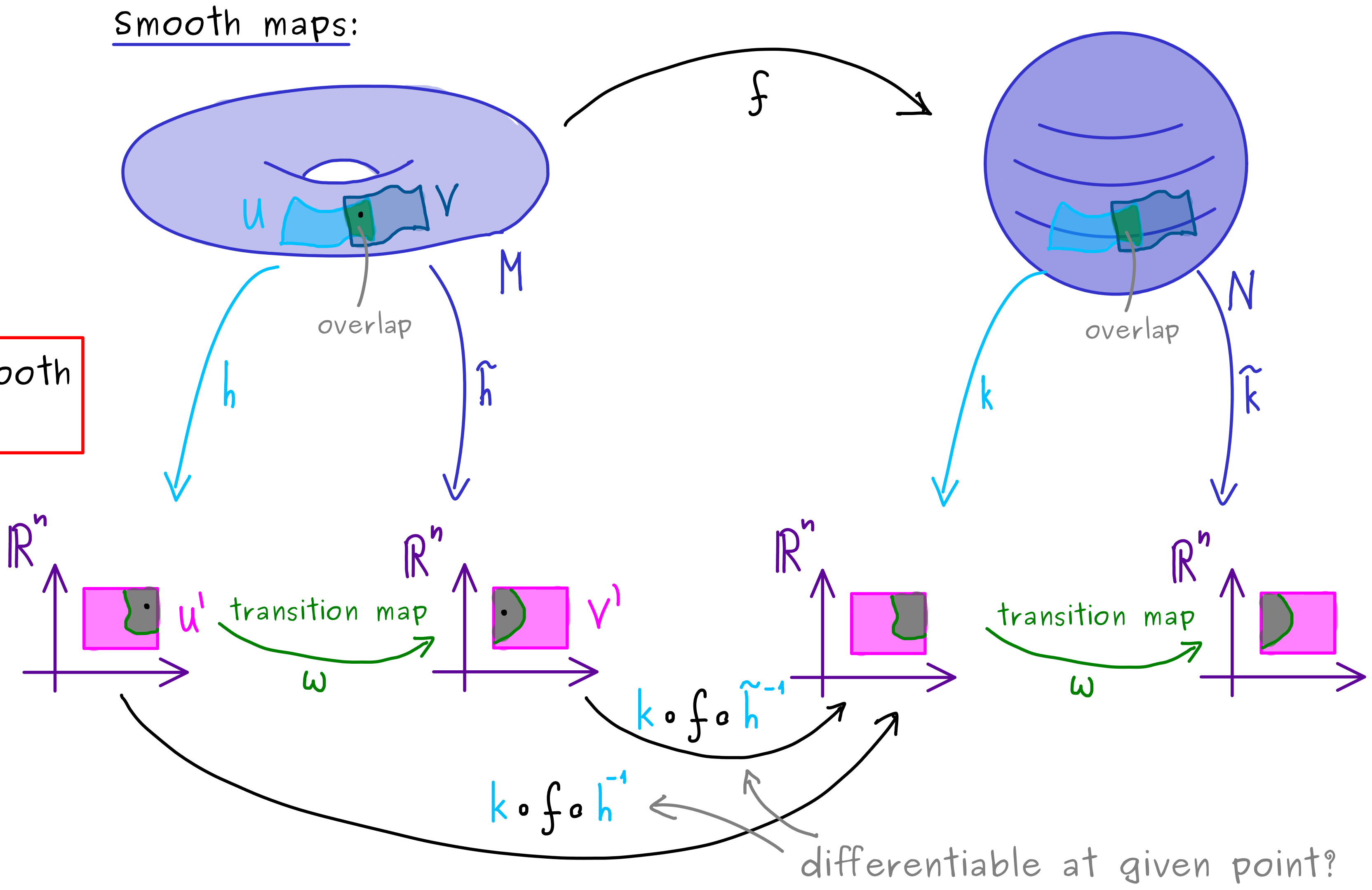
Hence: 1 is a regular value.

$\Rightarrow$ $f^{-1}[\{1\}] = S^{n-1}$ submanifold of $\mathbb{R}^n$.

Manifolds – Part 16

Smooth maps:

Use the smooth structures!



Definition: Let M and N be C^∞-smooth manifolds.

A map $f: M \longrightarrow N$ is called k-times differentiable at $p \in M$

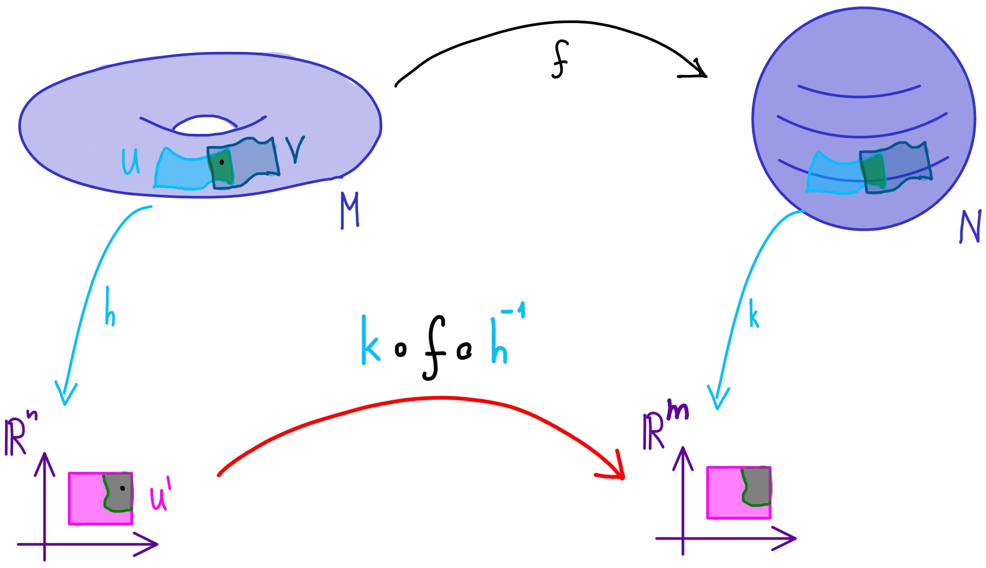
if for charts (U, h), (W, k) with $p \in U$ and $f(p) \in W$

the map $k \circ f \circ h^{-1}$ k-times differentiable at $h(p)$.

Moreover: $f: M \longrightarrow N$ is called C^∞-smooth if f is k-times differentiable at $p \in M$

for every $p \in M$ and every $k \in \mathbb{N}$. We write: $f \in C^\infty(M, N)$.

Manifolds – Part 17



$f: M \to N$

C^∞- smooth

Examples of smooth maps:

(1) $S^2 \longrightarrow \mathbb{R}^3$

inclusion map:

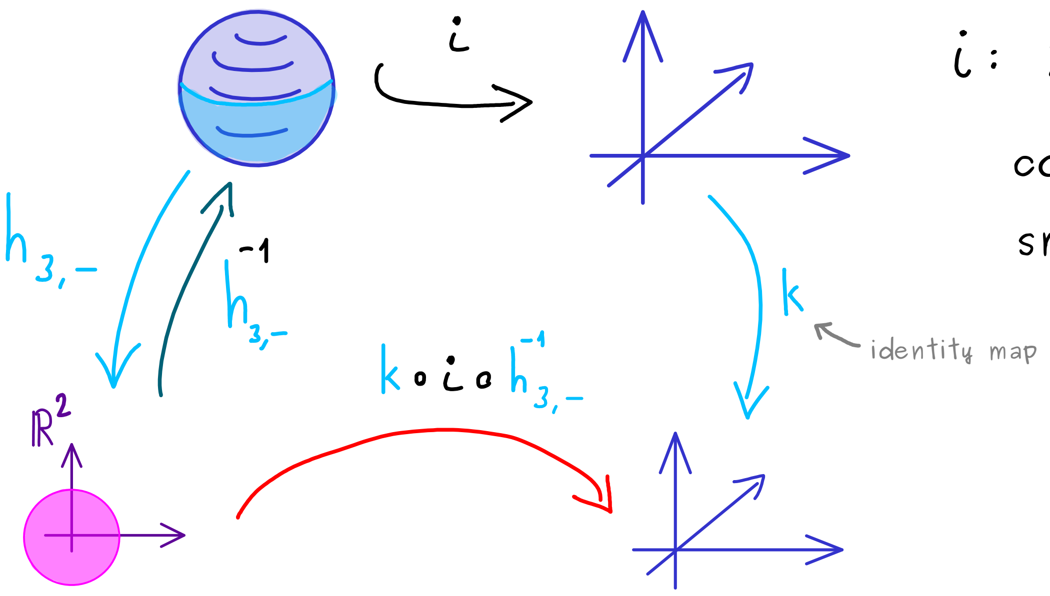
$i: x \longmapsto x$

continuous!

smooth?

$$h_{3,-}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

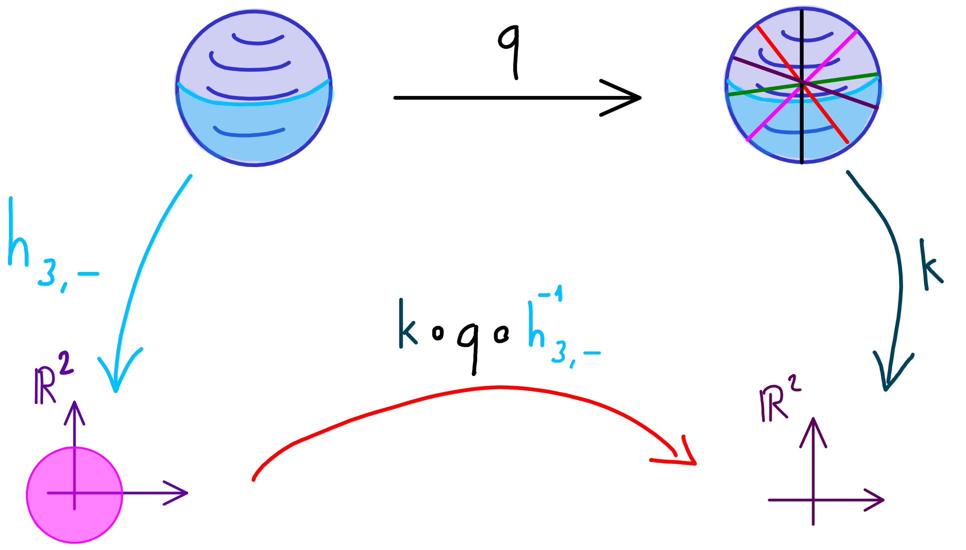
$$h^{-1}_{3,-}\left(\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\right) = \begin{pmatrix} x_1' \\ x_2' \\ -\sqrt{1-\|x'\|^2} \end{pmatrix}$$



$$k \circ i \circ h^{-1}_{3,-} : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \longmapsto \begin{pmatrix} x_1' \\ x_2' \\ -\sqrt{1-\|x'\|^2} \end{pmatrix}$$ differentiable $\Longrightarrow$ i is smooth

(2) $q : S^2 \longrightarrow P^2(\mathbb{R}) = S^2/\sim$ $\quad (x \sim y \Longleftrightarrow x = y \text{ or } x = -y)$

$x \longmapsto [x]_\sim$ continuous map! smooth?

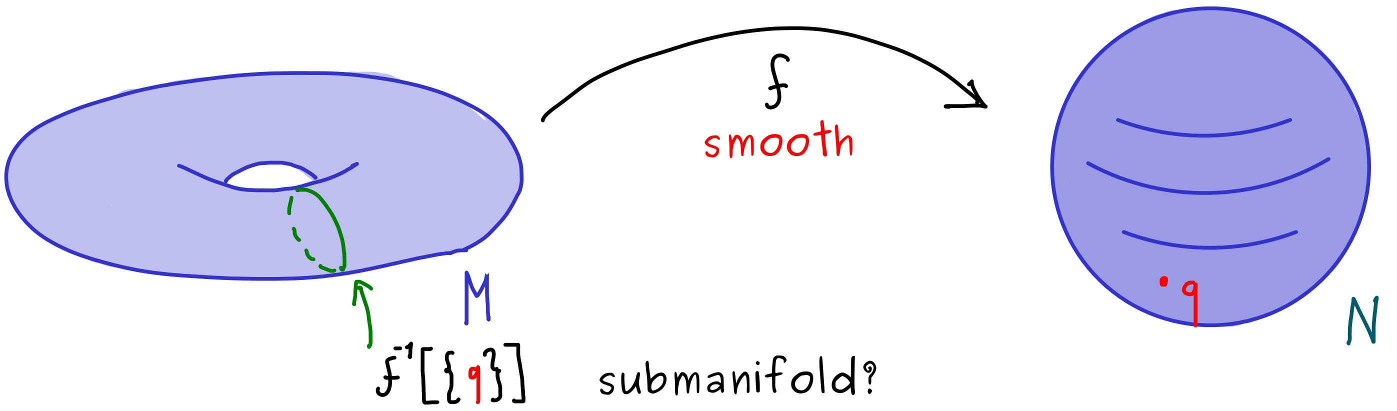


$$k\left(\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right]_\sim\right) = \begin{pmatrix} \frac{x_1}{x_3} \\ \frac{x_2}{x_3} \end{pmatrix}$$

$$\begin{tikzcd} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \arrow[r, mapsto, "h^{-1}_{3,-}"] \arrow[rrr, bend right, "C^\infty\text{- function}"'] & \begin{pmatrix} x'_1 \\ x'_2 \\ -\sqrt{1-\|x'\|^2} \end{pmatrix} \arrow[r, mapsto, "q"] & \left[\begin{pmatrix} x'_1 \\ x'_2 \\ -\sqrt{1-\|x'\|^2} \end{pmatrix}\right]_\sim \arrow[r, mapsto, "k"] & \begin{pmatrix} \frac{x'_1}{-\sqrt{1-\|x'\|^2}} \\ \frac{x'_2}{-\sqrt{1-\|x'\|^2}} \end{pmatrix} \end{tikzcd}$$

Manifolds – Part 18

Regular Value Theorem:



Let M, N be smooth manifolds of dimension m and n $(m \geq n)$,
$f: M \longrightarrow N$ be a smooth map, and $q \in N$ be a regular value of f.

↳ $f^{-1}[\{q\}]$ does not contain critical points

↳ $p \in M$ is called a critical point of f if

$$\operatorname{rank} f_p := \operatorname{rank}\left(J_{k\circ f\circ h^{-1}}(h(p))\right)$$

is less than n (not maximal!).

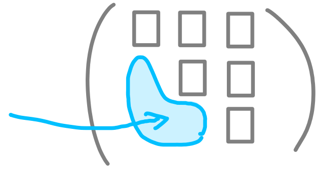
Then: $f^{-1}[\{q\}]$ is a $(m-n)$–dim submanifold of M.

Example:

(a) $GL(d,\mathbb{R}) := \{A \in \mathbb{R}^{d\times d} \mid \det(A) \neq 0\}$ is manifold of dimension d^2.

(b) $\operatorname{Sym}(d\times d,\mathbb{R}) := \{B \in \mathbb{R}^{d\times d} \mid B^T = B\}$ is manifold of dimension $\frac{d(d+1)}{2}$

$\frac{d^2-d}{2}$



$$d^2 - \frac{d^2-d}{2}$$

(c) $O(d,\mathbb{R}) := \{A \in GL(d,\mathbb{R}) \mid A^T A = \mathbb{1}\}$ is a submanifold of $GL(d,\mathbb{R})$

Proof: $f : GL(d,\mathbb{R}) \longrightarrow Sym(d\times d,\mathbb{R})\ ,\quad f(A) = A^T A$

Two things to show: (1) $f^{-1}[\{\mathbb{1}\}] = O(d,\mathbb{R})$

(2) $\mathbb{1}$ is a regular value of f

Case $d=2$:

$$\begin{tikzcd}
\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \arrow[d, mapsto] & GL(d,\mathbb{R}) \arrow[r, bend left, "f"] \arrow[d, "h"] & Sym(d\times d,\mathbb{R}) \arrow[d, "k"] & \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \arrow[d, mapsto] \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & \mathbb{R}^4 \arrow[r, "k\circ f\circ h^{-1}"] & \mathbb{R}^3 & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{tikzcd}$$

$$(k\circ f\circ h^{-1})\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (k\circ f)\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = k\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^T \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right)$$

$$= k\left(\begin{pmatrix} x_1^2 + x_3^2 & x_1 x_2 + x_3 x_4 \\ x_1 x_2 + x_3 x_4 & x_2^2 + x_4^2 \end{pmatrix}\right) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_1 x_2 + x_3 x_4 \\ x_2^2 + x_4^2 \end{pmatrix}$$

Jacobian matrix: $$J_{k\circ f\circ h^{-1}}(x) = \begin{pmatrix} 2x_1 & 0 & 2x_3 & 0 \\ x_2 & x_1 & x_4 & x_3 \\ 0 & 2x_2 & 0 & 2x_4 \end{pmatrix}$$

rank $= 3$? Not for: $x_1 = x_2 = 0$

$x_3 = x_4 = 0$

$x_1 = x_3 = 0$

$x_2 = x_4 = 0$

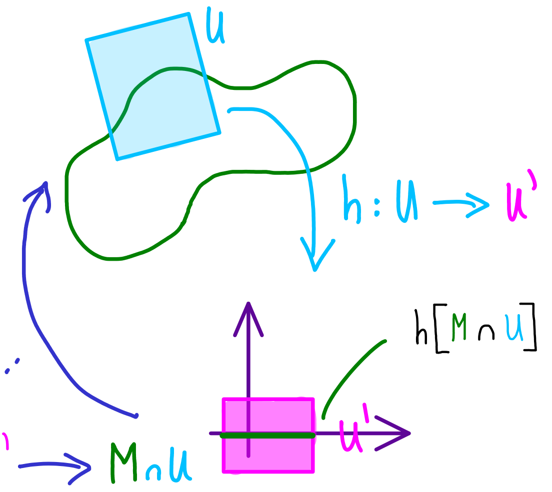
If $f(A) = \mathbb{1} \implies J_{k\circ f\circ h^{-1}}(h(A))$ has rank $3 \implies \mathbb{1}$ regular value

$\implies O(d,\mathbb{R})$ is a submanifold of dimension $d^2 - \dfrac{d(d+1)}{2} = \dfrac{d(d-1)}{2}$

Manifolds - Part 19

submanifold: $M \subseteq \mathbb{R}^n$ k-dimensional submanifold

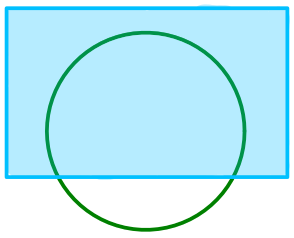
$$h[M \cap U] = (\mathbb{R}^k \times \underbrace{0}_{n-k \text{ zeros}}) \cap U'$$



$\varphi : \mathbb{R}^k \cap U' \longrightarrow M \cap U$

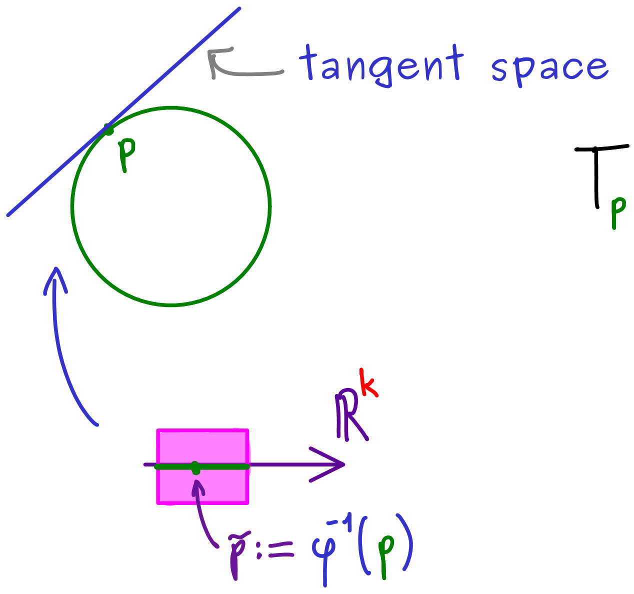
local parameterisation

Example:



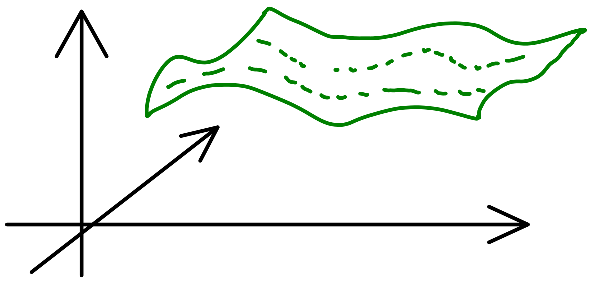
$$\varphi : \mathbb{R}^1 \cap U' \longrightarrow M \cap U$$
$$t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

Tangent space:



$$T_p^{\text{sub}} M := d\varphi_{\tilde{p}}[\mathbb{R}^k]$$
$$= \left\{ J_\varphi(\varphi^{-1}(p))\, x \;\middle|\; x \in \mathbb{R}^k \right\} \subseteq \mathbb{R}^n$$

Example:



surface given by a graph of a function:

$f : \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1(\mathbb{R}^2)$

$$M = G_f := \left\{ \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} \;\middle|\; (x,y) \in \mathbb{R}^2 \right\}$$

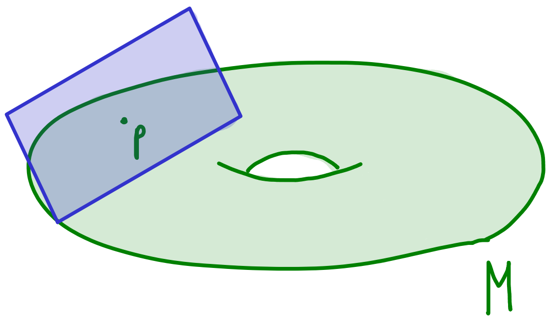
parameterisation: $\varphi : \mathbb{R}^2 \longrightarrow M \,, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

$$J_\varphi(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix}$$

$$\Rightarrow \underbrace{T_p^{\text{sub}}}_{p = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}} M = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x,y) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} \right)$$

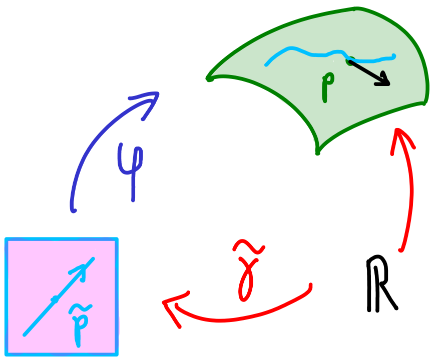
Manifolds – Part 20

$T_p^{sub} M$ tangent space for submanifold $M \subseteq \mathbb{R}^n$, $p \in M$



$$T_p^{sub} M := \left\{ J_\varphi(\varphi^{-1}(p))\, x \;\middle|\; x \in \mathbb{R}^k \right\} \subseteq \mathbb{R}^n$$

Idea:



parameterised curve $\gamma : \mathbb{R} \to M$

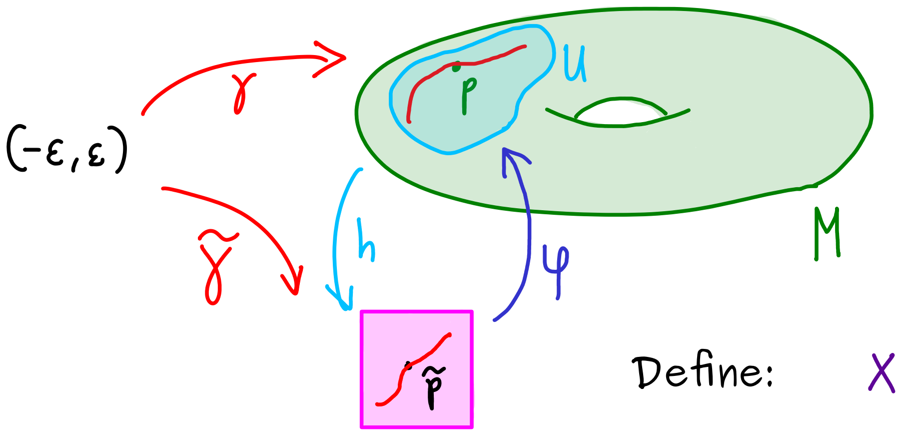
Proposition: $T_p^{sub} M = \left\{ \gamma'(0) \;\middle|\; \gamma : (-\varepsilon, \varepsilon) \to M \text{ differentiable with } \gamma(0) = p \right\}$

Proof: $(\subseteq)$ $v \in T_p^{sub} M \Rightarrow v = J_\varphi(\underbrace{\varphi^{-1}(p)}_{\tilde{p}})\, x$ for $x \in \mathbb{R}^k$, φ local parameterisation

$\Rightarrow v = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0)$ with $\tilde{\gamma}(t) = \tilde{p} + t x$, $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$

$= \frac{d}{dt} (\underbrace{\varphi \circ \tilde{\gamma}}_{\gamma}) \Big|_{t=0} = \gamma'(0)$

$(\supseteq)$ Take: $\gamma : (-\varepsilon, \varepsilon) \to M$ differentiable with $\gamma(0) = p$



Define: $X := \frac{d}{dt} (\underbrace{h \circ \gamma}_{\tilde{\gamma}}) \Big|_{t=0}$

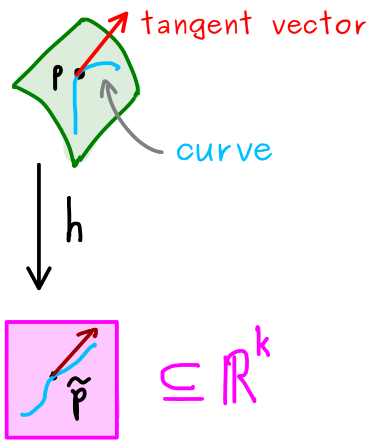
$$\gamma'(0) = \frac{d}{dt} (\varphi \circ \tilde{\gamma}) \Big|_{t=0} = J_\varphi(\tilde{\gamma}(0))\, \tilde{\gamma}'(0) = J_\varphi(\varphi^{-1}(p))\, X \in T_p^{sub} M$$

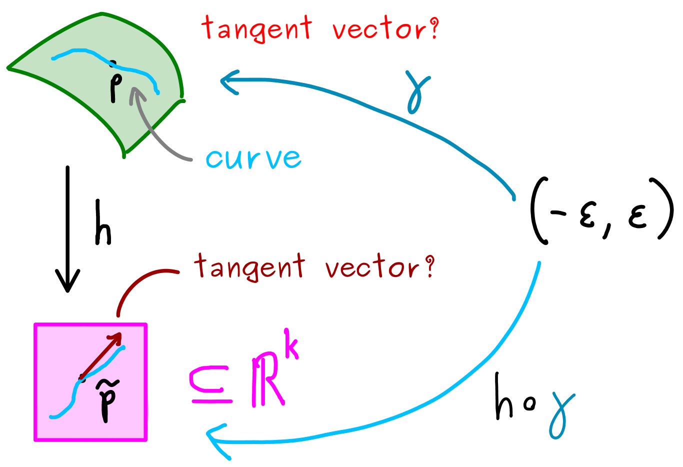
Manifolds – Part 21

$$T_p^{\text{sub}} M \rightsquigarrow T_p M$$

for $M \subseteq \mathbb{R}^n$ smooth submanifold

for M smooth manifold





Definition: $C_p(M) := \{ \gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma \text{ differentiable with } \gamma(0) = p \}$

$$\gamma \sim \alpha \;:\Longleftrightarrow\; (h \circ \gamma)'(0) = (h \circ \alpha)'(0)$$

for a chart (U, h).

equivalent class: $[\gamma]_\sim := \{ \alpha \mid \gamma \sim \alpha \}$ represents tangent vector

$$T_p M := C_p(M) / \sim$$

(set of all equivalence classes)

tangent space of the manifold M

Result:

- For a submanifold $T_p^{\text{sub}} M \underset{\text{bijection}}{\longleftrightarrow} T_p M$

  $\gamma'(0) \longleftrightarrow [\gamma]_\sim$

- $T_p M$ is a vector space with the operations:

  $v + w := h_*^{-1}\big(h_*(v) + h_*(w)\big)$ with $h_* : [\gamma]_\sim \mapsto \overset{\in \mathbb{R}^k}{(h \circ \gamma)'(0)}$

  $\lambda \cdot v := h_*^{-1}\big(\lambda \cdot h_*(v)\big)$

Manifolds – Part 22

smooth manifold M of dimension n, $p \in M$.

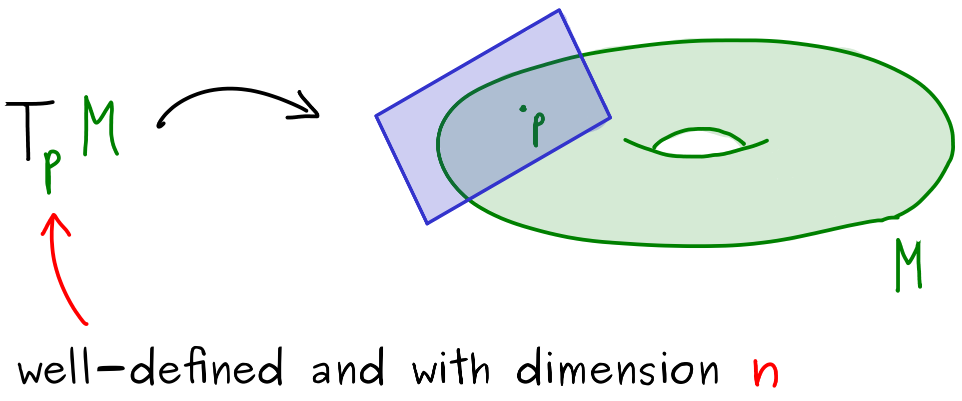
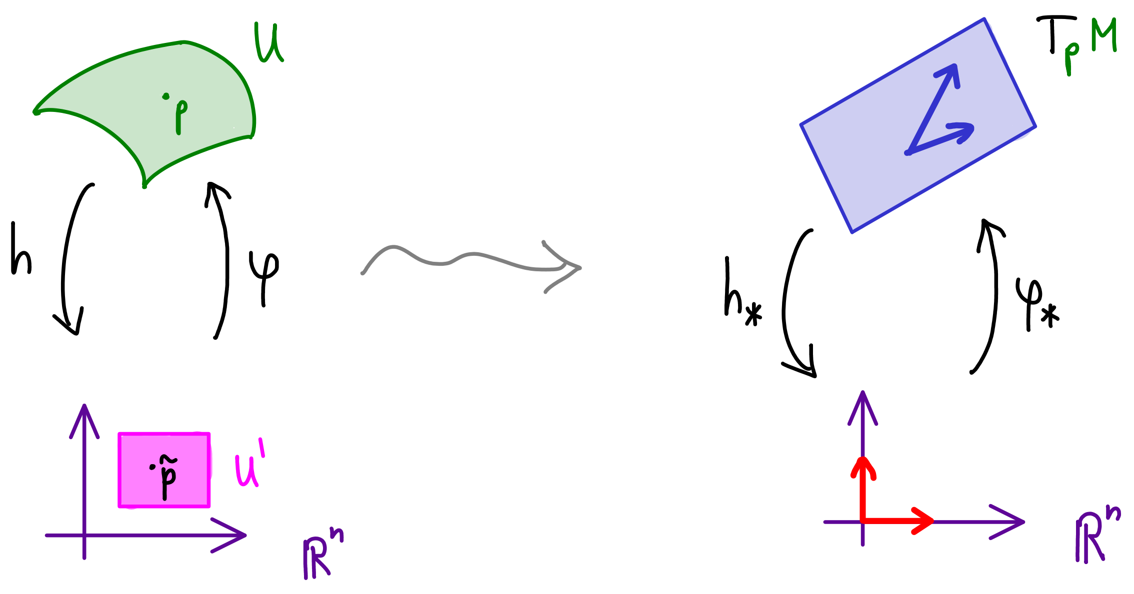


chart (U,h) :



defined by:

$h_* : T_p M \longrightarrow \mathbb{R}^n$

$[\gamma] \mapsto (h \circ \gamma)'(0)$

linear + bijective

$\varphi_* := h_*^{-1}$

Definition: coordinate basis (standard basis with respect to (U,h)) :

For (U,h) and $p \in U$, we define: $\partial_j := \varphi_*(e_j)$

where $(e_1, e_2, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$

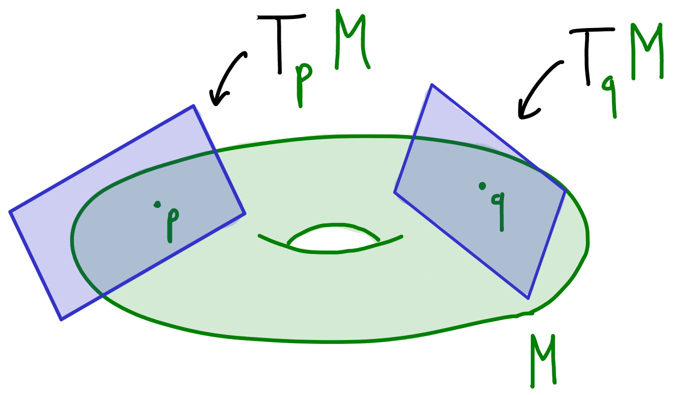
Remember: For submanifolds:

$$\begin{tikzcd}
T_p M \arrow[r] & T_p^{\text{sub}} M \\
\mathbb{R}^n \arrow[u, "\varphi_*"] \arrow[ur, "J_\varphi(\tilde{p})"'] &
\end{tikzcd}$$

$(\partial_1, \partial_2, \dots, \partial_n)$ is essentially $\left(\frac{\partial \varphi}{\partial x_1}(\tilde{p}), \frac{\partial \varphi}{\partial x_2}(\tilde{p}), \dots, \frac{\partial \varphi}{\partial x_n}(\tilde{p}) \right)$

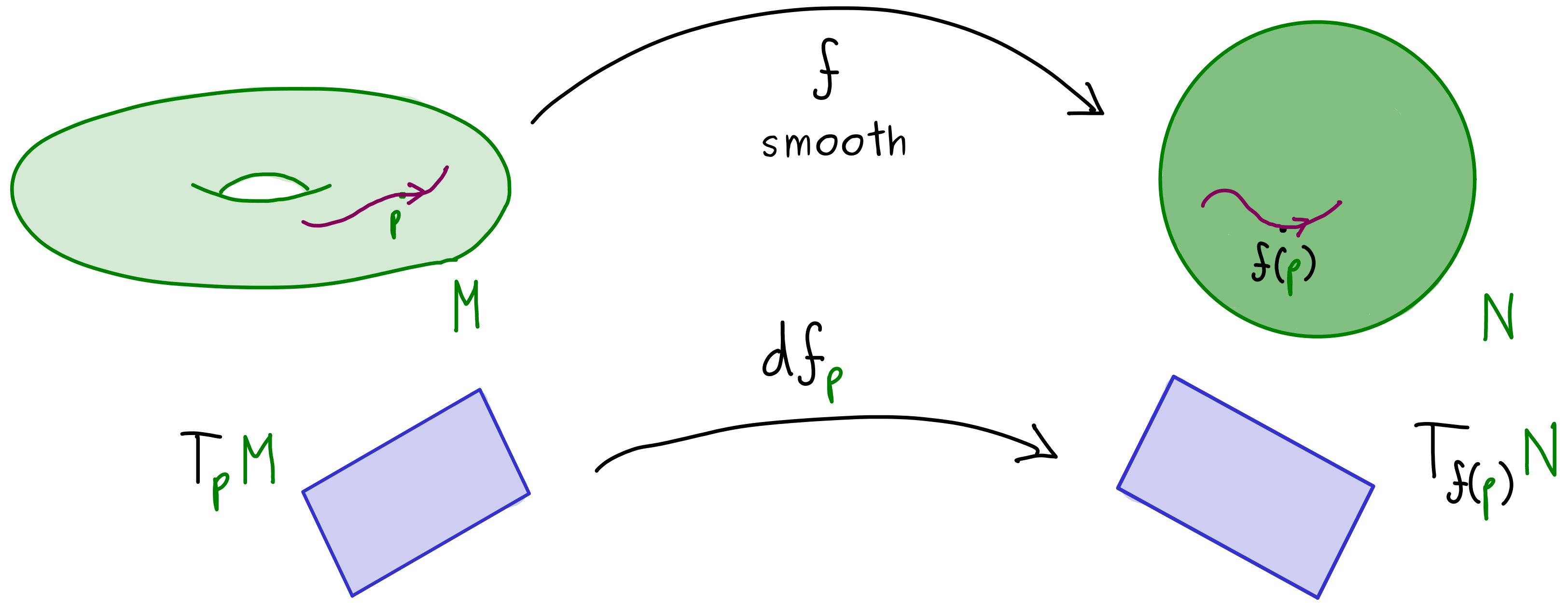
Soon: $f : M \longrightarrow N$ smooth $\rightsquigarrow$ $df_p : T_p M \longrightarrow T_p N$ differential

Manifolds – Part 23



Definition: tangent bundle $TM := \overset{\text{disjoint}}{\bigsqcup_{p\in M}} T_pM := \bigcup_{p\in M} \{p\}\times T_pM$

↳ smooth manifold of dimension $2\cdot \dim(M)$



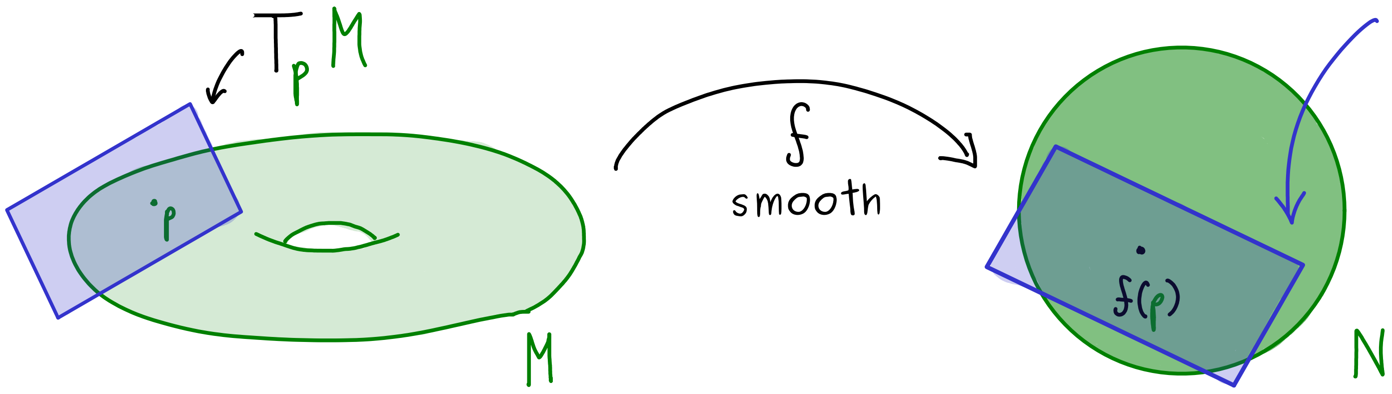
Definition: differential of f at point p

$$
\begin{aligned}
df_p : T_pM &\longrightarrow T_{f(p)}N \\
[\gamma] &\longmapsto [f\circ\gamma]
\end{aligned}
$$

differential: $df : p \longmapsto df_p$

Example for submanifolds $M, N \subseteq \mathbb{R}^n$ smooth submanifolds

$$T_{f(p)}N \overset{\text{bijection}}{=} T^{\text{sub}}_{f(p)}N$$



$$[\gamma] \stackrel{df_p}{\longmapsto} [f\circ\gamma] \overset{\text{bijection}}{=} (f\circ\gamma)'(0)$$

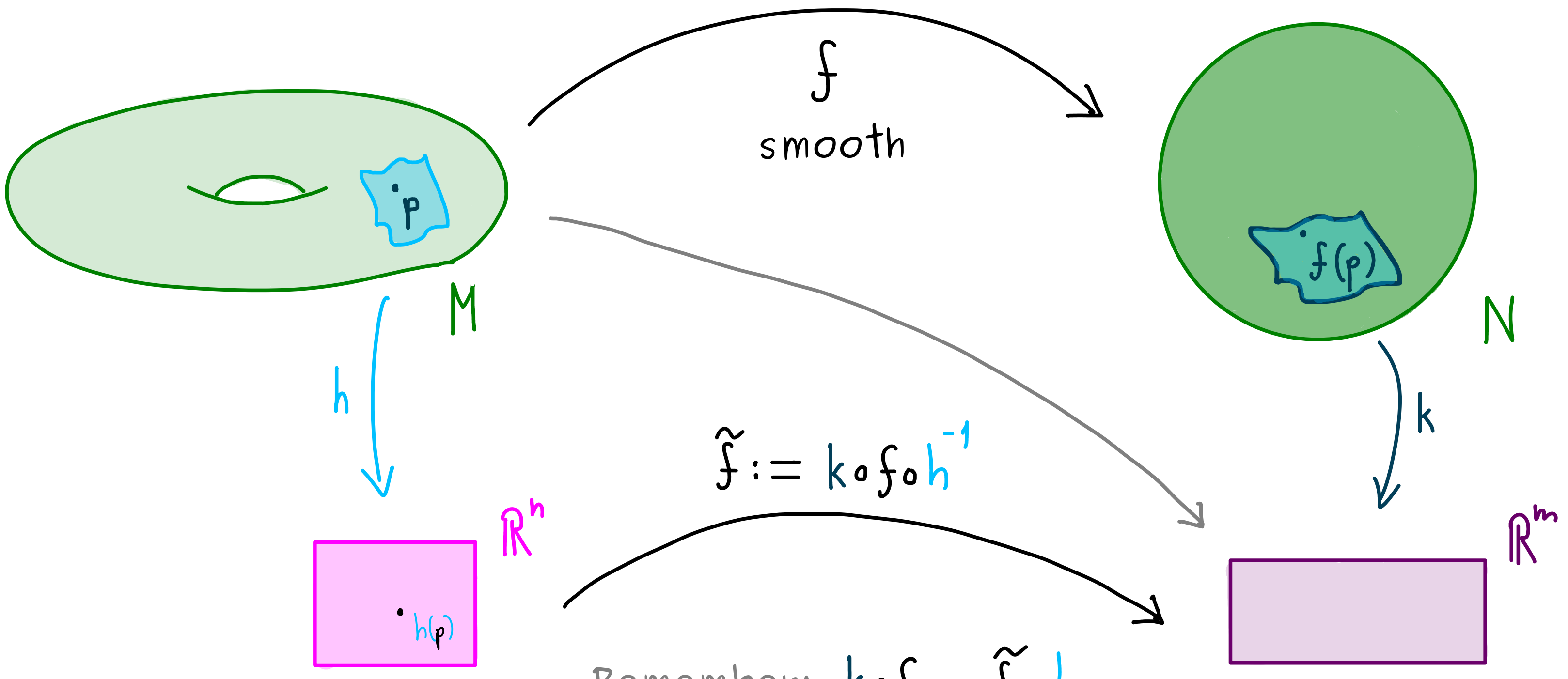
Example: $f: \mathbb{R}^n \to \mathbb{R}$ (smooth map)

$$df_p([\gamma]) \overset{\text{bijection}}{=} (f\circ\gamma)'(0) = J_f(\underbrace{\gamma(0)}_{p})\,\underbrace{\gamma'(0)}_{\text{tangent vector}}$$

$=$ directional derivative of f along $[\gamma]$ at p

Manifolds – Part 24

Differential in local charts?



Remember: $k\circ f = \tilde{f}\circ h$

Choose: $[\gamma]\in T_pM$:

$$dk_{f(p)}\big(df_p([\gamma])\big) = dk_{f(p)}\big([f\circ\gamma]\big)$$

$$= [k\circ f\circ\gamma] \overset{\text{bijection}}{=} (k\circ f\circ\gamma)'(0)$$

$$= (\tilde{f}\circ h\circ\gamma)'(0)$$

ordinary chain rule

$$= J_{\tilde{f}}(h(p))\,(h\circ\gamma)'(0)$$

$$\overset{\text{bijection}}{=} J_{\tilde{f}}(h(p))\,[h\circ\gamma]$$

$$= J_{\tilde{f}}(h(p))\,dh_p([\gamma])$$

Remember:

$$\boxed{\begin{aligned} f &= k^{-1}\circ\tilde{f}\circ h \\ df &= dk^{-1}\,J_{\tilde{f}}\,dh \end{aligned}}$$

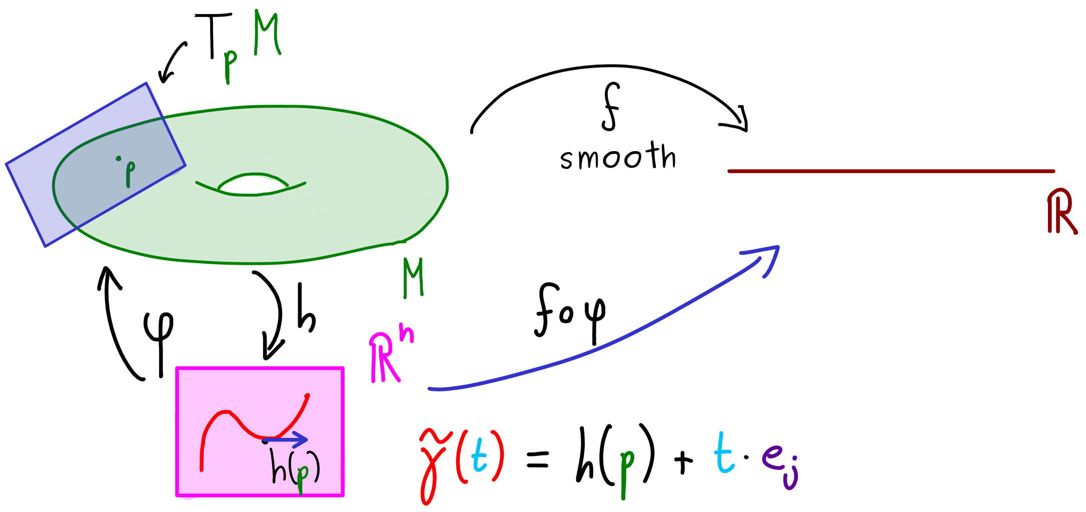
Manifolds - Part 25

Recall: $p\in M$, (U,h) : coordinate basis $(\partial_1,\dots,\partial_n)$ of T_pM

$$\varphi = h^{-1},\quad \partial_j := \varphi_*(e_j) = d\varphi_{h(p)}(e_j)$$

defined by:
$h_*: T_pM \to \mathbb{R}^n$
$[\gamma] \mapsto (h\circ\gamma)'(0)$
linear + bijective
$\varphi_* := h_*^{-1}$

Directional derivative: $f: M \to \mathbb{R}$ smooth



$\tilde\gamma(t) = h(p) + t\cdot e_j$

$$(\partial_j f)(p) := df_p(\partial_j)$$

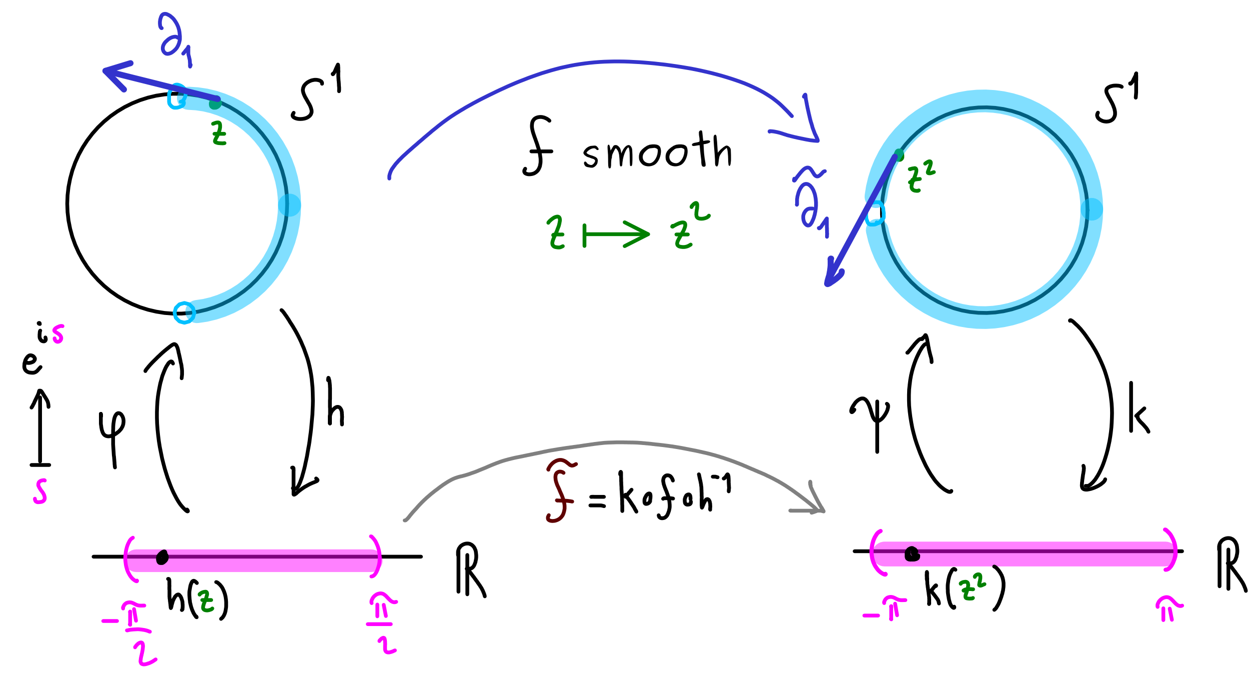
$$= df_p(d\varphi_{h(p)}(e_j))$$

$$= [f\circ\varphi\circ\tilde\gamma]$$

$$\overset{\text{bijection}}{=} (f\circ\varphi\circ\tilde\gamma)'(0)$$

$$\overset{\text{chain rule}}{=} J_{f\circ\varphi}(h(p))\,\underbrace{\tilde\gamma'(0)}_{e_j} = \frac{\partial(f\circ\varphi)}{\partial x_j}(h(p))$$

Example:



$$\partial_1 = d\varphi_{h(z)}(e_1) = [\varphi\circ\tilde\gamma],\quad \tilde\gamma(t) = \underbrace{h(z)}_{s} + t$$

$$= (\varphi\circ\tilde\gamma)'(0) = \frac{d}{dt}\Big|_{t=0} e^{i(s+t)} = i\cdot e^{is}$$

$$\tilde\partial_1 = d\psi_{k(f(z))}(e_1)$$

$$= (\psi\circ\tilde\gamma)'(0) \qquad \tilde\gamma(t) = \underbrace{k(z^2)}_{\substack{2s\\ \downarrow \\ (e^{is})^2}} + t$$

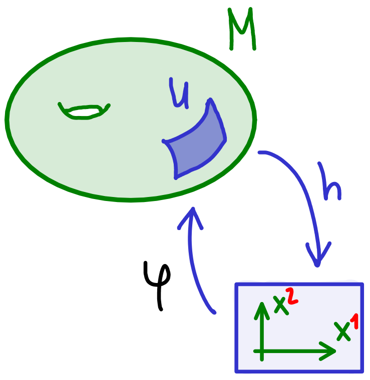
$$= i\cdot e^{2is}$$

map $\tilde f$: $s \overset{\varphi}{\mapsto} e^{is} \overset{f}{\mapsto} (e^{is})^2 \overset{k}{\mapsto} 2s$

$$J_{\tilde f}(s) = 2$$

differential of f: $df_z(\partial_1) \overset{\text{last video}}{=} dk_{z^2}^{-1}\,\underbrace{J_{\tilde f}(h(p))}_{2}\,\underbrace{dh_z(\partial_1)}_{e_1} = 2\cdot dk_{z^2}^{-1}(e_1) = 2\cdot\tilde\partial_1$

Manifolds - Part 26



Introduction to Ricci calculus / tensor calculus

↳ calculating in coordinates

↳ positions of indices matter

(superscripts, subscripts)

our language	Ricci calculus
components of a given chart (U,h) , $h: U \to \mathbb{R}^n$	$h^j : U \to \mathbb{R}$ coordinates or simply: $x^1, x^2, \dots, x^n$
coordinate basis of T_pM : $\partial_j := \varphi_*(e_j)$	$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}$
tangent vector $[\gamma] \in T_pM$: $v_1\partial_1 + v_2\partial_2 + \dots + v_n\partial_n$	$v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n} =: v^j \frac{\partial}{\partial x^j}$ (Einstein summation convention) contravariant vector
Later: inner product on T_pM : $\langle v, w \rangle \in \mathbb{R}$	$v^j \underbrace{g_{jk}} w^k$ → tensor

dual to a contravariant vector: $v_j \underbrace{dx^j}$

↳ one-form (⇝ linear map)

$$dx_j(\partial_k) = \begin{cases} 1 \ , & j = k \\ 0 \ , & j \neq k \end{cases}$$

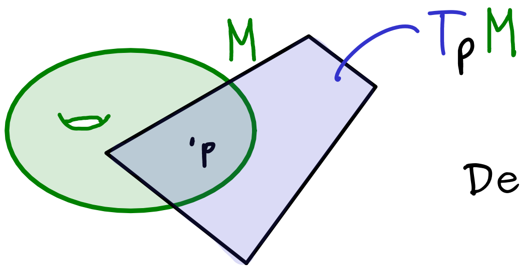
$$= \delta_{jk}$$

Kronecker delta

$$dx^j\left(\frac{\partial}{\partial x^k}\right) = \delta^j{}_k$$

Manifolds - Part 27

Recall:



T_pM n-dimensional vector space

Define: $T_p^*M := (T_pM)^*$

$= \{\alpha : T_pM \to \mathbb{R} \text{ linear}\}$

$\rightsquigarrow dx_{j,p} : T_pM \to \mathbb{R}$

$dx_{j,p}(\partial_k) = \delta_{jk}$ linear map!

differential form: map ω defined on M such that $\omega(p) \in T_p^*M$
(one-form)

$dx_j : p \mapsto dx_{j,p} \in T_p^*M$

Some multilinear algebra:

$$\text{Alt}^k(V) := \{\alpha : \underbrace{V \times \cdots \times V}_{k\text{-times}} \longrightarrow \mathbb{R} \text{ multilinear } (k\text{-linear}) + \text{alternating}\}$$

alternating: $\alpha(v_1, \dots, v_k) = 0$ if $(v_1, \dots, v_k)$ linearly dependent

Example: $\alpha \in \text{Alt}^2(V)$, $\alpha(v_1, v_2) = -\alpha(v_2, v_1)$

$\det \in \text{Alt}^2(\mathbb{R}^2)$

$\alpha \in \text{Alt}^k(V)$ is called an alternating k-form on V

Remember: $\text{Alt}^1(V) = V^*$ (dual space of V)

$\text{Alt}^0(V) = \mathbb{R}$

Manifolds - Part 28

Wedge product: $\wedge$ multiplication defined for $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$

$$\wedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^s(V) \longrightarrow \mathrm{Alt}^{k+s}(V)$$

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta$$

(k+s)-linear

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) :\neq \alpha(v_1, \dots, v_k) \cdot \beta(v_{k+1}, \dots, v_{k+s})$$

not a possible definition!
(not alternating)

Definition: For $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^s(V)$, we define $\alpha \wedge \beta \in \mathrm{Alt}^{k+s}(V)$ by:

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+s}) := \frac{1}{k! \cdot s!} \sum_{\sigma \in S_{k+s}} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+s)})$$

Examples: (a) $\alpha, \beta \in \mathrm{Alt}^1(V) = V^*$:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

(b) $\alpha, \beta \in \mathrm{Alt}^1(\mathbb{R}^3)$, $\alpha\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1$, $\beta\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_2 = \underbrace{(0, 1, 0)}_{\text{identified with } \beta} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$(\alpha \wedge \beta)\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 y_2 - y_1 x_2 = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{identified with } \alpha \wedge \beta} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

Properties:

(a) $\alpha \wedge \beta = (-1)^{k \cdot s} \beta \wedge \alpha$ (anticommutative)

(b) $(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta$

$(\lambda \alpha) \wedge \beta = \lambda (\alpha \wedge \beta)$ (bilinear)

(c) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (associative)

(d) For a linear map $f: W \to V$ and $\alpha \in Alt^k(V)$ define:

pullback $(f^* \alpha)(w_1, \dots, w_k) := \alpha(f(w_1), \dots, f(w_k))$

("natural")

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$$

Manifolds - Part 29

M smooth manifold of dimension n $\Rightarrow$ T_pM n-dimensional vector space

Definition:

$$\omega : M \longrightarrow \bigcup_{p \in M} \mathrm{Alt}^k(T_pM)$$

$$p \longmapsto \omega_p = \omega(p) \in \mathrm{Alt}^k(T_pM)$$

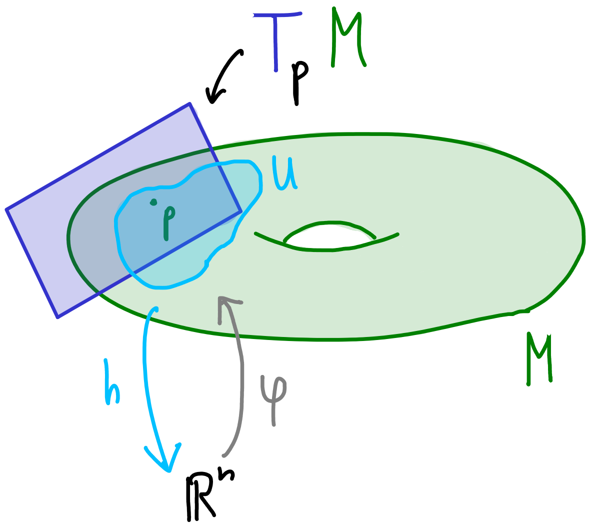
is called a k-form on M.

We also define: $\omega \wedge \eta$ as $(\omega \wedge \eta)(p) := \omega(p) \wedge \eta(p)$

$f^* \omega$ as $(f^*\omega)(p) := (df_p)^* \omega(f(p))$

$f : N \longrightarrow M$ smooth

Basis elements:



basis of T_pM : $(\partial_1, \partial_2, \dots, \partial_n)$ with $\partial_j := \varphi_*(e_j) = d\varphi_{h(p)}(e_j)$

basis of $(T_pM)^* = \mathrm{Alt}^1(T_pM)$: $(dx_p^1, dx_p^2, \dots, dx_p^n)$

defined by: $dx_p^j(\partial_k) = \delta^j_k = \begin{cases} 1 , & j = k \\ 0 , & j \neq k \end{cases}$

Proposition: A basis of $\mathrm{Alt}^k(T_pM)$ is given by:

$$\left(dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \dots \wedge dx_p^{\mu_k} \right)_{\mu_1 < \mu_2 < \dots < \mu_k}$$

Example: $\dim(M) = 3$, $\mathrm{Alt}^2(T_pM)$:

$$\left(dx_p^1 \wedge dx_p^2 ,\ dx_p^1 \wedge dx_p^3 ,\ dx_p^2 \wedge dx_p^3 \right)$$

Conclusion: Each k-form on M can locally be written as:

$$\omega(p) = \sum_{\mu_1 < \cdots < \mu_k} \omega_{\mu_1,\mu_2,\ldots,\mu_k}(p) \cdot dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \cdots \wedge dx_p^{\mu_k}$$

$\omega_{\mu_1,\mu_2,\ldots,\mu_k} : \mathcal{U} \longrightarrow \mathbb{R}$ component functions

Definition:

- If all component functions are differentiable at p, then ω is differentiable at p.
- If ω is differentiable at all $p \in M$, then ω is called a differential form on M.

$\omega \in \Omega^k(M)$

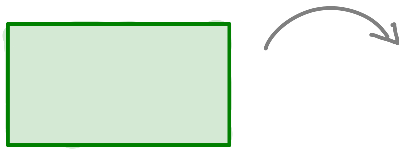
$$\Omega^0(M) := C^\infty(M)$$

Manifolds — Part 30

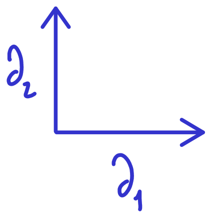
differential form on a manifold: $\omega \in \Omega^k(M) \leftarrow$ k-form on M + differentiable

$$\omega(p) = \sum_{\mu_1 < \cdots < \mu_k} \omega_{\mu_1, \mu_2, \ldots, \mu_k}(p) \cdot dx_p^{\mu_1} \wedge dx_p^{\mu_2} \wedge \cdots \wedge dx_p^{\mu_k}$$

Examples: (a) $M = \mathbb{R}^2$



$T_p M =$



$\partial_k = e_k$

$$dx_p^j(\partial_k) = \delta^j{}_k$$

identify: $\partial_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $dx_p^1 = (1, 0)$

$\partial_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $dx_p^2 = (0, 1)$

$$(dx_p^1 \wedge dx_p^2)(\underset{\begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}}{\underset{\shortparallel}{a_1}}, \underset{\begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}}{\underset{\shortparallel}{a_2}}) = \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)\, dx_p^1(a_{\sigma(1)})\, dx_p^2(a_{\sigma(2)})$$

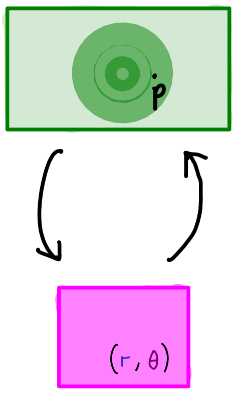
$$= \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

(b) Each $\omega \in \Omega^n(\mathbb{R}^n)$ can be written as:

$$\omega(p) = \omega_{1,2,\ldots,n}(p)\; dx_p^1 \wedge dx_p^2 \wedge \cdots \wedge dx_p^n$$

$$= \omega_{1,2,\ldots,n}(p) \det\left(\vdots \; \vdots \; \cdots \; \vdots \right)$$

(c) $M = \mathbb{R}^2$



φ given by polar coordinates $\varphi(r, \theta) = \begin{pmatrix} r \cdot \cos(\theta) \\ r \cdot \sin(\theta) \end{pmatrix}$

$$\partial_j := \varphi_*(e_j) = J_\varphi(\tilde{p})(e_j)$$

$$\partial_1(r, \theta) = \frac{\partial \varphi}{\partial r}(r, \theta) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$$\partial_2(r, \theta) = \frac{\partial \varphi}{\partial \theta}(r, \theta) = \begin{pmatrix} -r \cdot \sin(\theta) \\ r \cdot \cos(\theta) \end{pmatrix}$$

corresponding 1-forms: $dr_p = \left(\cos(\theta) \, , \, \sin(\theta) \right) = \frac{1}{\sqrt{x^2+y^2}} (x, y)$

for $p = (x,y)$ $\quad d\theta_p = \frac{1}{r} \left(-\sin(\theta) \, , \, \cos(\theta) \right) = \frac{1}{x^2+y^2} (-y, x)$

2-form: $(dr_p \wedge d\theta_p)(e_1, e_2) = dr_p(e_1)\, d\theta_p(e_2) - dr_p(e_2)\, d\theta_p(e_1)$

$$= \frac{1}{r} \left(\cos(\theta) \right)^2 - \frac{1}{r} \cdot (-1) \left(\sin(\theta) \right)^2$$

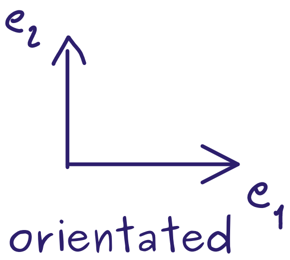
$$= \frac{1}{r}$$

$$\Longrightarrow \quad r \, dr_p \wedge d\theta_p = \det\left(\begin{array}{cc} \vdots & \vdots \end{array} \right) = dx_p \wedge dy_p$$

Manifolds – Part 31

vector space $\leftsquigarrow$ orientation

for example: $\mathbb{R}^n$ with basis: $\mathcal{B} = (e_1, e_2, \dots, e_n)$ ← positively orientated



change-of-basis matrix → $T_{\mathcal{C} \leftarrow \mathcal{B}}$

$\mathcal{C} = (c_1, c_2, \dots, c_n)$

two cases:

$\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$: positively orientated

$\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) < 0$: negatively orientated

$\Longrightarrow$ two equivalence classes for bases

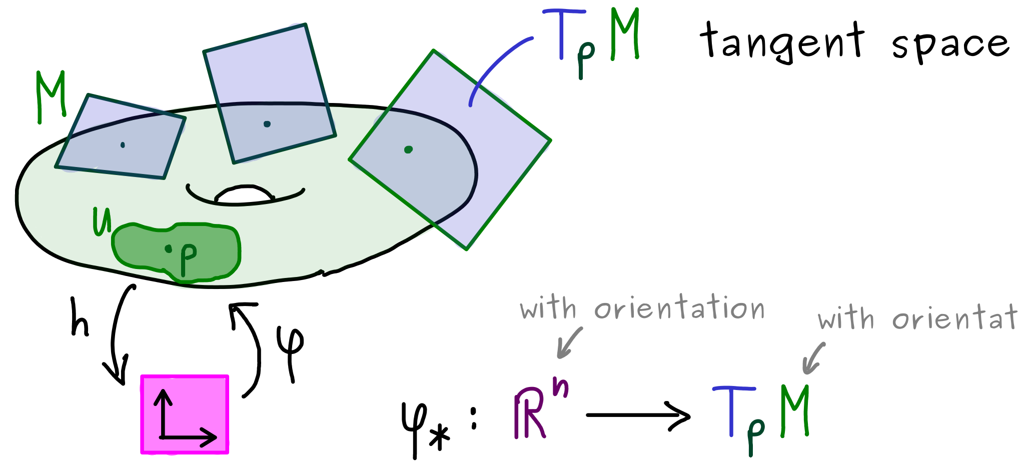
$[\{\text{basis}\}]$ $\circlearrowleft$ $\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$

$[\{\text{basis}\}]$ $\circlearrowleft$ $\det(T_{\mathcal{C} \leftarrow \mathcal{B}}) > 0$

Remember: V finite-dimensional vector space + one chosen equivalence class

$\rightsquigarrow$ orientation (V, or)

Orientations for manifolds:



with orientation ↓ $\quad$ with orientation ↓

$$\varphi_* : \mathbb{R}^n \longrightarrow T_p M, \quad e_j \longmapsto \partial_j^{(h)}(p)$$

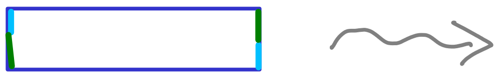
↑ should not change orientation

Definition: A smooth manifold M is called orientable if there is a family of orientations for the tangent spaces $\{(T_pM, or_p)\}$ such that

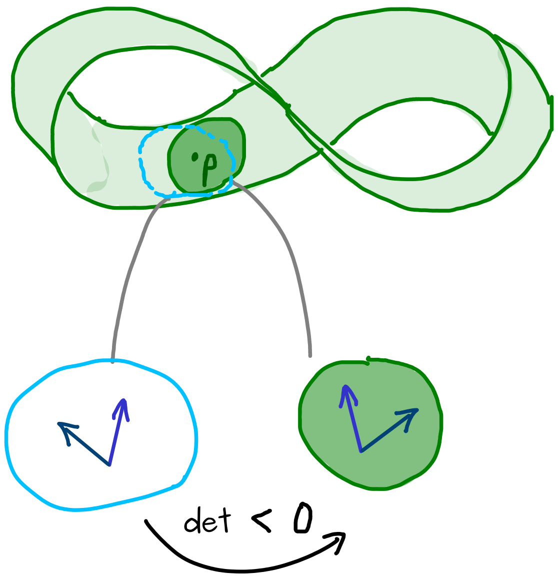
$$\forall p \in M \;\; \exists (U,h) \;\; \forall x \in U : \quad \left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \ldots, \partial_n^{(h)}(x)\right) \in or_x$$

Example: (a) If M has an atlas with one chart (M,h), then M is orientable.

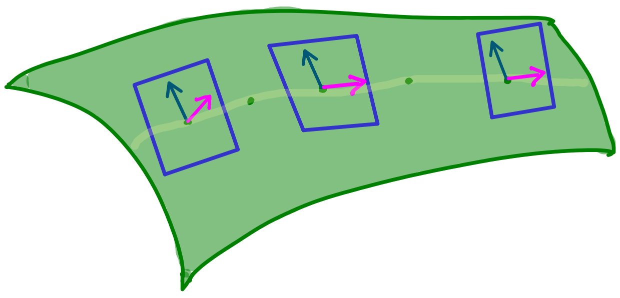
(b) Möbius strip:



after running around the strip:



Manifolds - Part 32



orientable manifold M

Fact: Let M be an n-dim smooth manifold. Then the following claims are equivalent:

(a) M is orientable : We have $\{ (T_p M , or_p) \}$ such that

$$\forall p \in M \;\; \exists (U,h) \;\; \forall x \in U : \;\; \left(\partial_1^{(h)}(x) , \partial_2^{(h)}(x) , \dots , \partial_n^{(h)}(x) \right) \in or_x$$

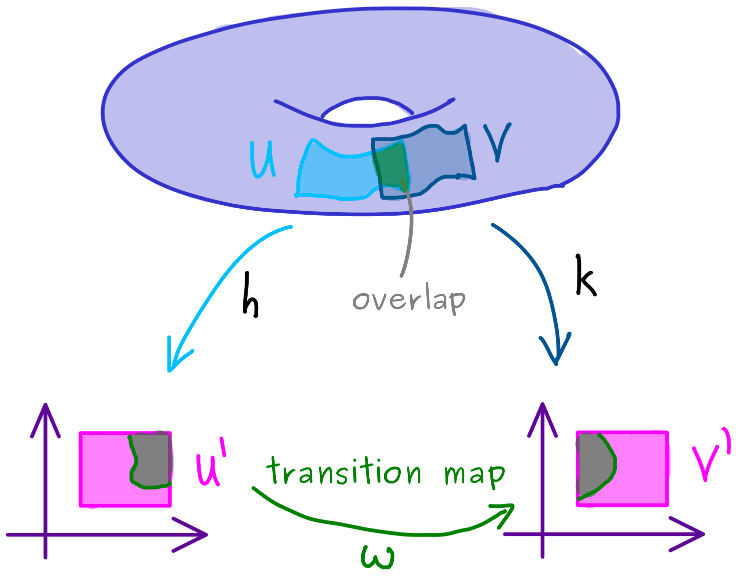
(b) There is an atlas for M

collection of charts that cover the manifold

such that all transition maps

$\omega :$  $\longrightarrow$  satisfy:

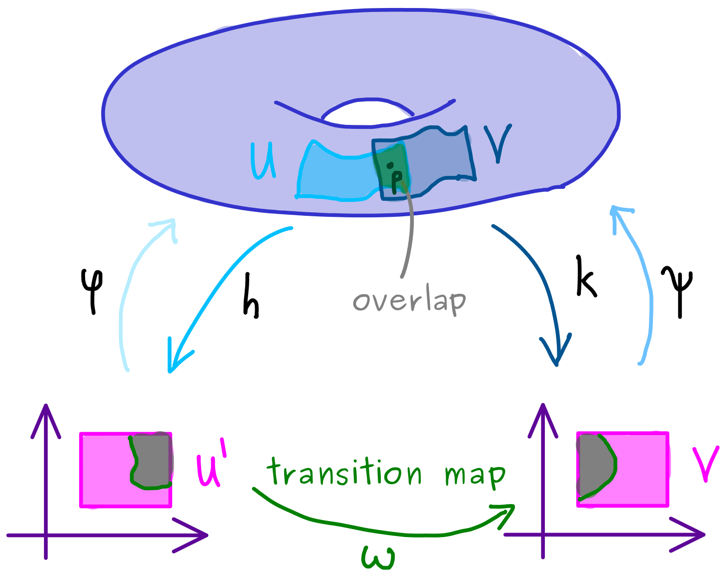
$$\det (J_\omega (x)) > 0$$



(c) There is a differential form (volume form)

$$\omega \in \Omega^n(M) \quad \text{with} \quad \omega(p) \neq 0 \quad \text{for all } p \in M .$$

Proof: (a) $\Longleftrightarrow$ (b)



$\rightsquigarrow$

$$\begin{tikzcd}
 & T_pM \arrow[dl, bend right, "dh_p"'] \arrow[dr, bend left, "dk_p"] & \\
\mathbb{R}^n_{\tilde p} \arrow[ur, bend right, "\varphi_*"'] \arrow[rr, bend right, "J_\omega(\tilde p)"] & & \mathbb{R}^n \arrow[ul, bend left, "\psi_*"]
\end{tikzcd}$$

We have: $\psi_*(\underbrace{J_\omega(\tilde p)\, e_1}_{\text{first column of Jacobian}}) = \varphi_*(e_1)$

$$= \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} = \sum_j \lambda_j e_j$$

$$\Longrightarrow \sum_{j=1}^{n} \lambda_j \underbrace{\psi_*(e_j)}_{\partial_j^{(k)}(p)} = \underbrace{\varphi_*(e_1)}_{\partial_1^{(h)}(p)} \quad (*)$$

Change-of-basis matrix: $\mathcal{B} = \left(\partial_1^{(h)}(p), \dots, \partial_n^{(h)}(p)\right) \underset{T_{\mathcal{C}\leftarrow\mathcal{B}}}{\rightsquigarrow} \mathcal{C} = \left(\partial_1^{(k)}(p), \dots, \partial_n^{(k)}(p)\right)$

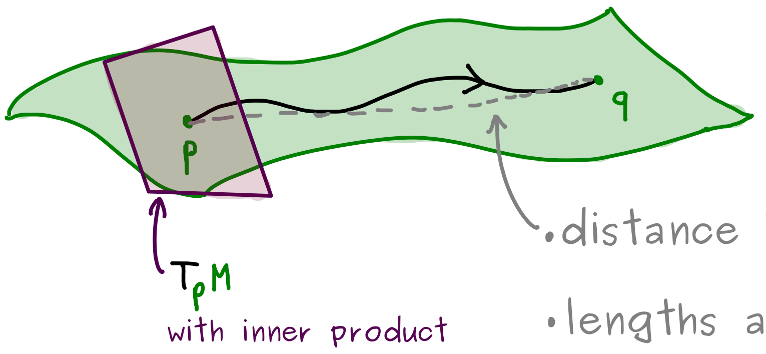
$$\overset{(*)}{\Longrightarrow} \quad T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \lambda_1 & \cdots \\ \lambda_2 & \\ \vdots & \\ \lambda_n & \end{pmatrix} = J_\omega(\tilde p)$$

Hence:

$$\det(T_{\mathcal{C}\leftarrow\mathcal{B}}) > 0 \quad \Longleftrightarrow \quad \det(J_\omega(x)) > 0$$

$$(a) \quad \Longleftrightarrow \quad (b)$$

Manifolds — Part 33

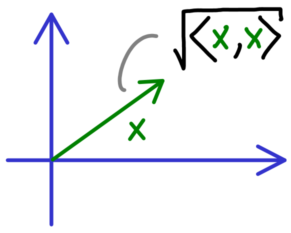


manifold M

- distance between p and q ?
- lengths and angles?

In $\mathbb{R}^n$: inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

write: $g(x,y) = \langle x,y\rangle$



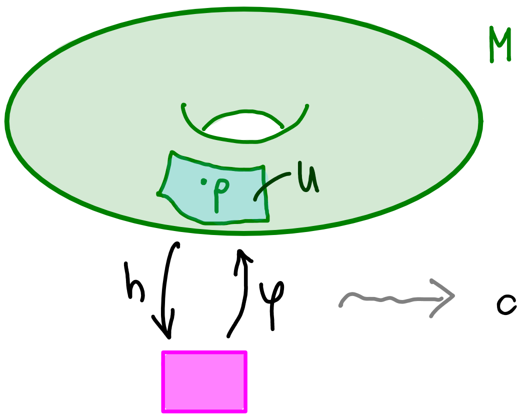
length of x

Definition: M smooth manifold. If we have an inner product g_p on T_pM for all $p\in M$ and $p \mapsto g_p$ smooth, then:

$g: p \mapsto g_p$ is called a Riemannian metric and

(M,g) is called a Riemannian manifold.

What does smooth mean?



$\rightsquigarrow$ coordinate basis in T_xM , $x\in U$

$$\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right)$$

$$g_x\left(\partial_i^{(h)}(x), \partial_j^{(h)}(x)\right) =: g_{ij}^{(h)}(x)$$

maps: $U \longrightarrow \mathbb{R}$ smooth!

$x \longmapsto g_{ij}^{(h)}(x)$

for all i,j , (U,h)

In local coordinates:

$$g_x(\cdot,\circ) \overset{\substack{\text{(Einstein summation convention)}\\ \downarrow}}{=} g_{ij}^{(h)}(x)\, dx_x^i(\cdot)\, dx_x^j(\circ)$$

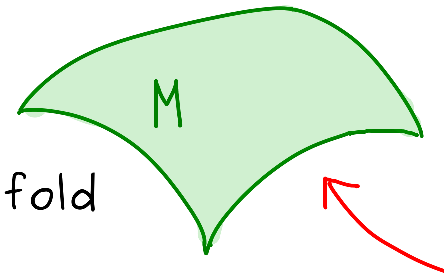
Hence: g_x can be seen as a symmetric matrix: $G = \left(g_{ij}^{(h)}(x)\right)_{ij}$

Manifolds - Part 34

Riemannian metric: $g : p \mapsto g_p$ inner product on $T_p M$

smooth

Submanifolds in $\mathbb{R}^N$:

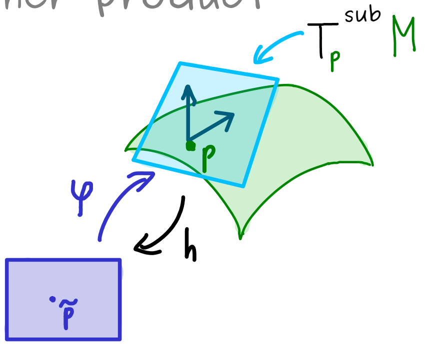


$\subseteq \mathbb{R}^N$

n-dimensional submanifold

standard inner product

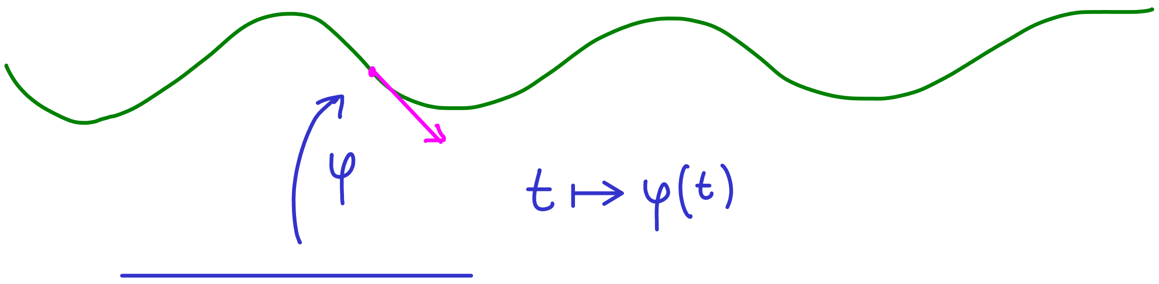
standard Riemannian metric



Note: $T_p M \cong T_p^{\text{sub}} M = \text{Span}\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}\right)$

$$g_{ij}^{(h)}(p) = \left\langle \frac{\partial \varphi}{\partial x_i}(\tilde{p}), \frac{\partial \varphi}{\partial x_j}(\tilde{p}) \right\rangle_{\text{standard}}$$

Examples: (a) 1-dimensional submanifold in $\mathbb{R}^N$



$t \mapsto \varphi(t)$

$$g_{11}^{(h)}(p) = \langle \varphi'(t), \varphi'(t) \rangle_{\text{standard}} = \|\varphi'(t)\|^2_{\text{standard}}$$

length: $\displaystyle\int_a^b \|\varphi'(t)\|_{\text{standard}} \, dt = \int_a^b \sqrt{\det(G)} \, dt$

(b) $S^2 \subseteq \mathbb{R}^3$ has parameterization given by spherical coordinates:

$$\Phi(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

$\Longrightarrow$ two tangent vectors: $\frac{\partial \Phi}{\partial \theta} = \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{pmatrix}$

$$\frac{\partial \Phi}{\partial \varphi} = \begin{pmatrix} -\sin(\theta)\sin(\varphi) \\ \sin(\theta)\cos(\varphi) \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \rightsquigarrow \sqrt{\det(G)} = |\sin(\theta)|$$

volume form: $\sqrt{\det(G)}\, d\theta \wedge d\varphi$

Manifolds — Part 35

We already know: An orientable n-dimensional manifold M has a non-trivial volume form $\omega \in \Omega^n(M)$.

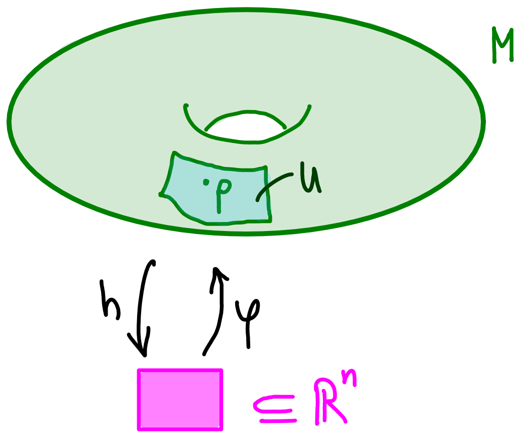
Definition: M orientable Riemannian manifold of dimension n.

Then the canonical volume form $\omega_M \in \Omega^n(M)$ is defined by:

If $(v_1, v_2, \dots, v_n)$ is a positively orientated basis of T_pM and an orthonormal basis of T_pM (ONB), $g_p(v_i, v_j) = \delta_{ij}$

then: $\omega_M(p)(v_1, v_2, \dots, v_n) = 1$

Proposition: (M, g) orientable Riemannian manifold of dimension n.



Let (U, h) be a chart such that the basis

$$\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right)$$

is positively orientated for all $x \in U$.

dual basis

$$\omega_M(x) = \sqrt{\det(G)}\, dx_x^1 \wedge dx_x^2 \wedge \dots \wedge dx_x^n$$

where $G_{ij} := g_x\left(\partial_i^{(h)}(x), \partial_j^{(h)}(x)\right)$

determinant of Gram/ Gramian

Proof:

$$\begin{tikzcd} \left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right) \arrow[r, bend left, "\text{Gram-Schmidt}"] & \left(v_1, v_2, \dots, v_n\right) \arrow[l, bend left, "f"] \end{tikzcd} \quad \text{ONB}$$

positively orientated

f linear map

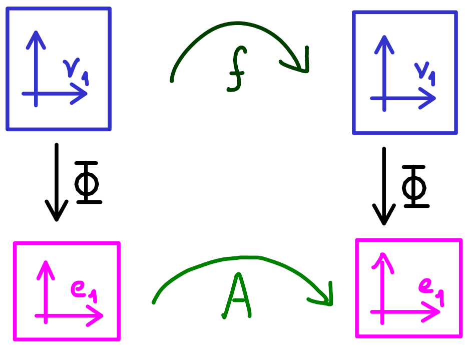
positively orientated

Then: $$\omega_M(x)\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right)$$

$$= \omega_M(x)\left(f(v_1), f(v_2), \dots, f(v_n)\right) = f^*\omega_M(x)(v_1, \dots, v_n)$$

$$= \det(f)\, \underbrace{\omega_M(x)(v_1, \dots, v_n)}_{=1}$$

$$g_x\left(\partial_i^{(h)}(x), \partial_j^{(h)}(x)\right) = g_x\left(f(v_i), f(v_j)\right)$$



$$= g_x\left(\Phi^{-1} A \Phi(v_i), \Phi^{-1} A \Phi(v_j)\right)$$

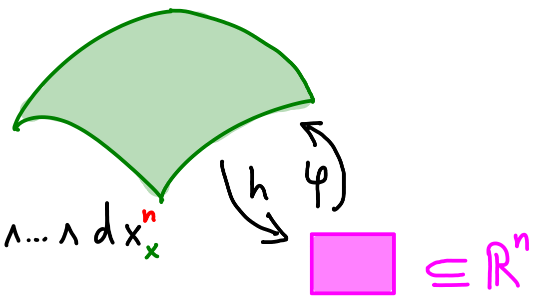
$$= \left\langle A \underbrace{\Phi(v_i)}_{e_i}, A \underbrace{\Phi(v_j)}_{e_j} \right\rangle_{\text{standard}} = \left(A^T A\right)_{ij}$$

$$\Longrightarrow \quad \det(G) = \det(A)^2 \qquad \square$$

Manifolds - Part 36

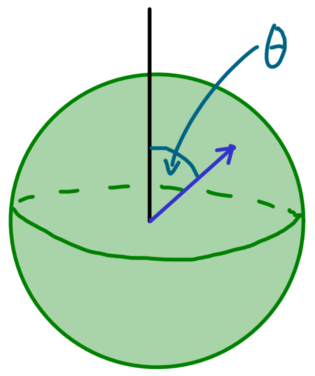
M orientable Riemannian manifold of dimension n.

↳ canonical volume form $\omega_M(x) = \sqrt{\det(G)}\; dx^1_x \wedge \dots \wedge dx^n_x$



Examples: (a) $S^2 \subseteq \mathbb{R}^3$ has parameterization given by spherical coordinates:

$$\Phi(\theta,\varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

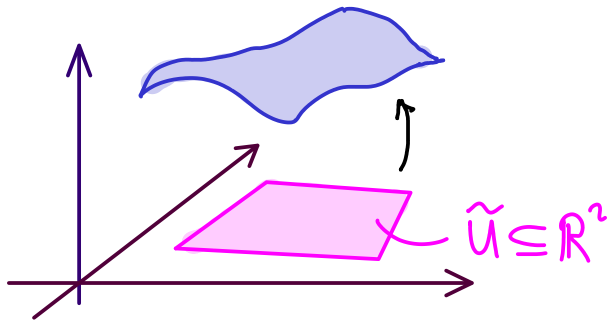


$$\Longrightarrow \quad G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}$$

$$\Longrightarrow \quad \omega_M(x) = \sin(\theta)\; d\theta \wedge d\varphi$$

(b) Graph surface: $f: \mathbb{R}^2 \to \mathbb{R}$ C^∞-function

$$M := \{ (x, f(x)) \mid x \in \mathbb{R}^2 \}$$



2-dim. submanifold in $\mathbb{R}^3$

Use parameterization: $\varphi: x \mapsto (x, f(x))$, $h: (x, f(x)) \mapsto x$

tangent vectors: $\partial_1^{(h)}(p) \overset{\text{identify}}{=} \dfrac{\partial \varphi}{\partial x_1}(x) = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x_1}(x) \end{pmatrix}$

$$\partial_2^{(h)}(p) \overset{\text{identify}}{=} \frac{\partial \varphi}{\partial x_2}(x) = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x_2}(x) \end{pmatrix}$$

$$g_{ij}^{(h)}(p) = \left\langle \frac{\partial\varphi}{\partial x_i}(x), \frac{\partial\varphi}{\partial x_j}(x) \right\rangle_{\text{standard}} = \begin{cases} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, & i \neq j \\ 1 + \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} & i = j \end{cases}$$

$$\Longrightarrow \quad G = \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x_1}\right)^2 & \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2} & 1 + \left(\frac{\partial f}{\partial x_2}\right)^2 \end{pmatrix}$$

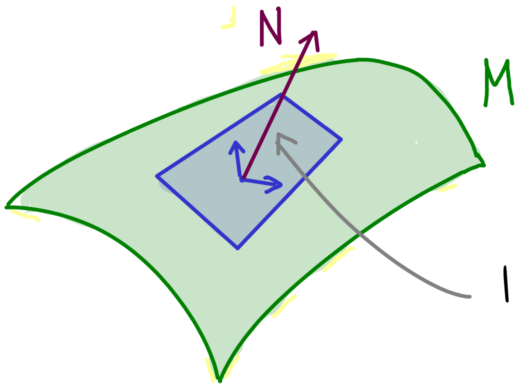
$$\det(G) = 1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2$$

Canonical volume form: $\omega_M(p) = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2} \; dx_p^1 \wedge dx_p^2$

Interesting fact: $\left\| \partial_1^{(h)}(p) \times \partial_2^{(h)}(p) \right\|_{\text{standard}} = \left\| \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x_1}(x) \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x_2}(x) \end{pmatrix} \right\|_{\text{standard}}$

$$= \left\| \begin{pmatrix} -\frac{\partial f}{\partial x_1} \\ -\frac{\partial f}{\partial x_2} \\ 1 \end{pmatrix} \right\|_{\text{standard}} = \sqrt{\det(G)}$$

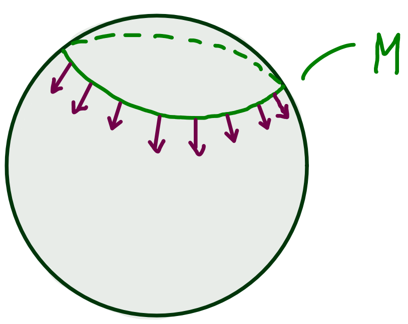
Manifolds – Part 37



$M \subseteq \mathbb{R}^3$ orientable Riemannian manifold of dimension 2

length of N $\leftrightsquigarrow$ canonical volume form

Definition: Let $\tilde{M}$ be a Riemannian manifold and $M \subseteq \tilde{M}$.



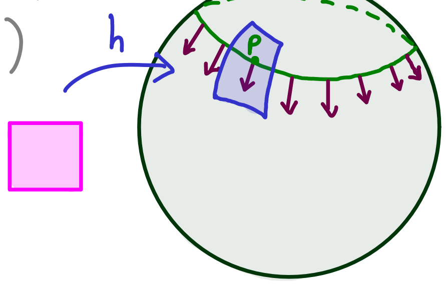
A map
$$N : M \longrightarrow T\tilde{M}$$
$$p \longmapsto N(p) \in T_p\tilde{M}$$

and $N(p) \in (T_pM)^{\perp} \setminus \{0\}$ (see $T_pM \subseteq T_p\tilde{M}$)

(orthogonal w.r.t. g_p)

is called a normal vector field.

We call it continuous at p if for a chart (U,h) of $\tilde{M}$ $(p \in U)$ holds:



$$N(x) = \sum_i a_i(x) \cdot \partial_i^{(h)}(x)$$

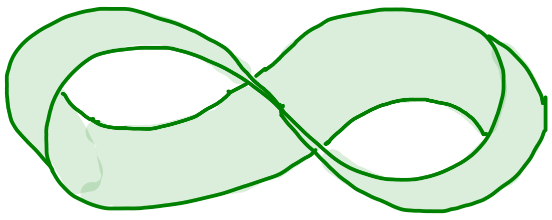
continuous functions $U \to \mathbb{R}$

We call it a continuous unit normal vector field if

- N is continuous at every $p \in M$
- $\|N(x)\| = \sqrt{g_x(N(x),N(x))} = 1$ for all $x \in M$.

Important fact: $M \subseteq \mathbb{R}^n$ $(n-1)$-dimensional submanifold:

(a) M is orientable $\Longleftrightarrow$ M has a continuous unit normal vector field



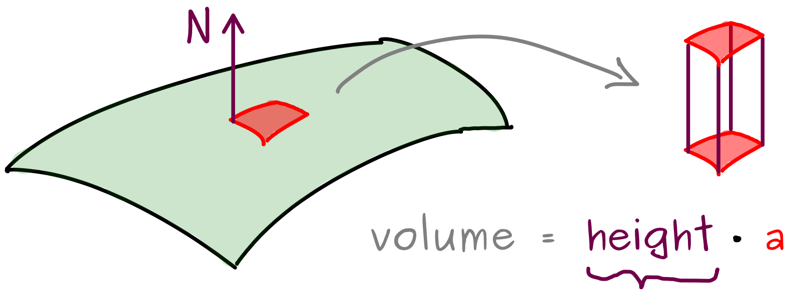
continuous normal vector field not possible

(b) If N is a continuous unit normal vector field, then:

canonical volume form → $$\omega_M = N \lrcorner \det$$

means:

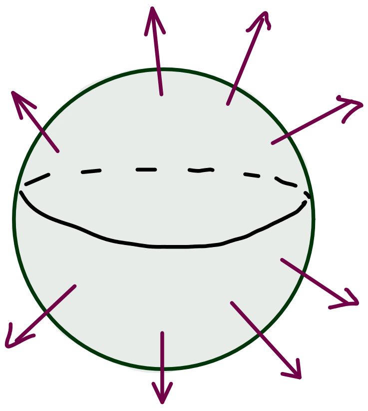
$$\omega_M(x)(v_1, \dots, v_{n-1}) = \det\big(N(x), v_1, \dots, v_{n-1}\big)$$



$$\text{volume} = \underbrace{\text{height}}_{=1} \cdot \text{area}$$

Example: $S^2 \subseteq \mathbb{R}^3$,

$N(x) = x$



parameterization:

$$\Phi(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

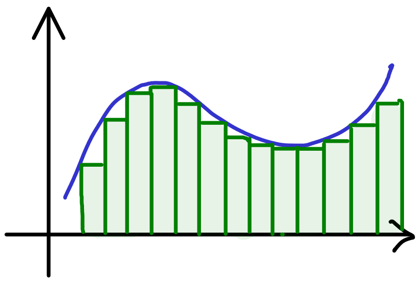
$$\sqrt{\det(G)} = \omega_M(x)\big(\partial_1^{(h)}(x), \partial_2^{(h)}(x)\big) = \det\big(N(x), \partial_1^{(h)}(x), \partial_2^{(h)}(x)\big)$$

$$= \det \begin{pmatrix} \sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & \sin(\theta)\cos(\varphi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{pmatrix}$$

$$= \sin(\theta)$$

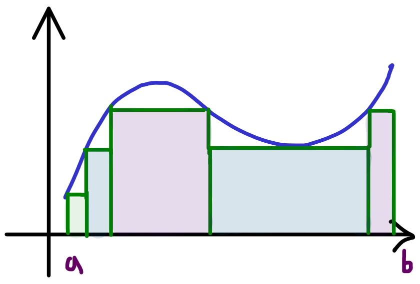
Manifolds - Part 38

Integration: $f: \mathbb{R} \longrightarrow \mathbb{R}$ (smooth function later)



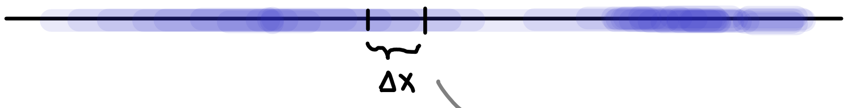
Riemann integral

$$\int_{[a,b]} f(x)\, dx$$



Lebesgue integral

See $f(x)$ as a density at point $x \in \mathbb{R}$:

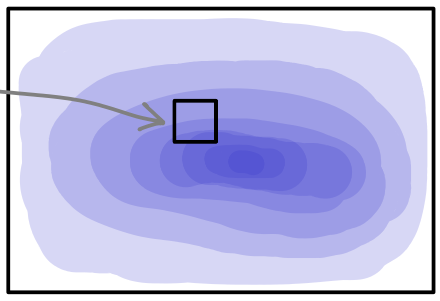


$$f(x) \cdot \Delta x$$

density · length = mass

$$\sum f(x) \cdot \Delta x \rightsquigarrow \int_{\mathbb{R}} f(x)\, dx = \text{total mass}$$

Same idea in higher dimensions: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



density · area = mass

$$\rightsquigarrow \int_{\mathbb{R}^2} f(x,y)\, d(x,y) = \text{total mass}$$

Let's take $M = \mathbb{R}^2$: differential form $\omega : p \mapsto f(p)\, dx \wedge dy \in \mathrm{Alt}^2(\underbrace{T_pM}_{=\mathbb{R}^2})$

$$\rightsquigarrow \quad \omega_p(v,w) = f(p)\left(\underbrace{dx(v)}_{v_1} \cdot \underbrace{dy(w)}_{w_2} - \underbrace{dx(w)}_{w_1} \cdot \underbrace{dy(v)}_{v_2} \right)$$

$$= f(p)\, \det(v, w)$$

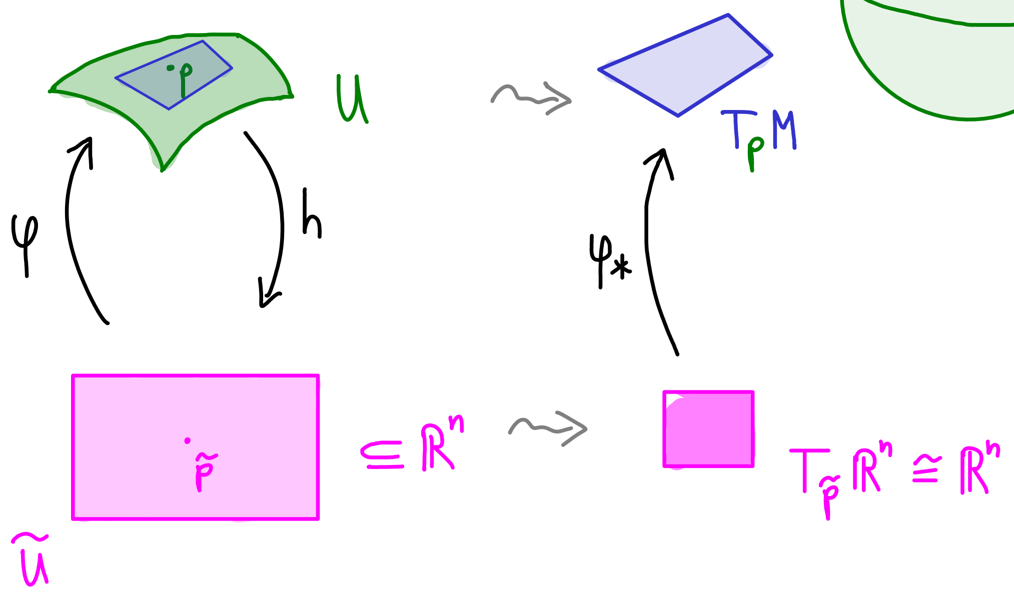
integral: $\displaystyle\int_M \omega := \int_M f\, dx \wedge dy = \int_{\mathbb{R}^2} f(x,y)\, d(x,y)$

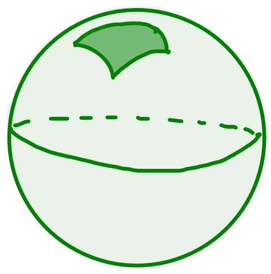
Manifolds – Part 39

Integration on $\mathbb{R}^n$: $\int_{\mathbb{R}^2} f(x,y)\; d(x,y) \;=:\; \int_{\mathbb{R}^2} f \; dx \wedge dy$

$$= -\int_{\mathbb{R}^2} f \; dy \wedge dx$$

Integration on orientable manifolds:



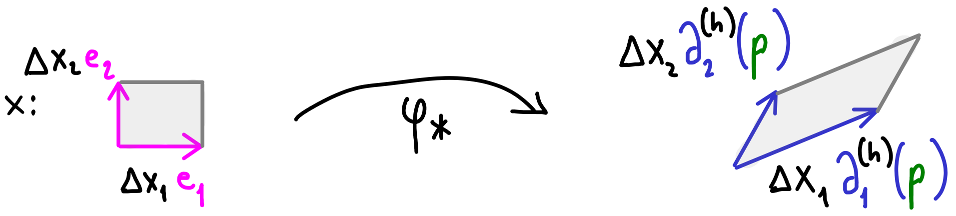


(U,h) chart with:

$(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x))$

is positively orientated for all $x \in U$.

Consider small box:



volume: $\Delta x_1 \cdot \Delta x_2 \cdots \Delta x_n$

measured by ω_p

$$\omega_p\big(\Delta x_1 \cdot \partial_1^{(h)}(p),\, \Delta x_2 \cdot \partial_2^{(h)}(p), \dots, \Delta x_n \cdot \partial_n^{(h)}(p)\big)$$

$$= \underbrace{\omega_p\big(\partial_1^{(h)}(p), \dots, \partial_n^{(h)}(p)\big)}_{= \;\omega_{1,2,\dots,n}(p)} \cdot \Delta x_1 \cdots \Delta x_n$$

$$\xrightarrow[\text{limit process}]{\text{summing up small boxes}} \qquad \int_{\tilde{U}} \omega_{1,2,\dots,n}(\varphi(\tilde{p}))\; dx_1\, dx_2 \cdots dx_n$$

Definition: Let M be an orientable n-dimensional manifold, $\omega \in \Omega^n(M)$, (U,h) chart with: $\left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \ldots, \partial_n^{(h)}(x)\right)$ is positively orientated for all $x \in U$.

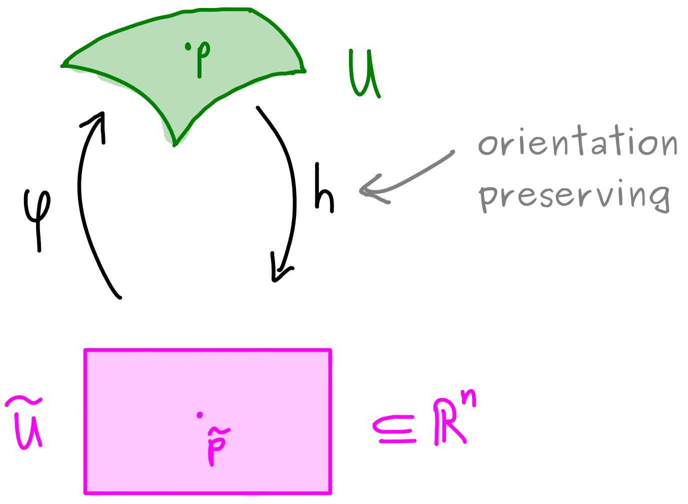
For $A \subseteq U$, where $h[A]$ is measurable, we define:

$$\int_A \omega := \int_{h[A]} \omega_{1,2,\ldots,n}\left(h^{-1}(x)\right) dx$$

Manifolds - Part 40

Let M be an orientable n-dimensional manifold and $\omega \in \Omega^n(M)$.

$$\omega(p) = \underbrace{\omega_{1,2,\dots,n}(p)}_{\text{component function}} \, dx_p^1 \wedge dx_p^2 \wedge \dots \wedge dx_p^n$$



$$\int_U \omega := \int_{h[U]} \omega_{1,2,\dots,n}(h^{-1}(x)) \, dx \qquad \text{(integral in } \mathbb{R}^n\text{)}$$

$$= \int_{h[U]} \varphi^* \omega$$

volume form on manifold $\mathbb{R}^n$

Some explanations:

(1) For $\omega \in \Omega^n(U)$, $\varphi : \tilde{U} \longrightarrow U$, we define $\varphi^*\omega \in \Omega^n(\tilde{U})$

by: $$(\varphi^*\omega)_{\tilde{p}}(v_1, \dots, v_n) := \underset{(p = \varphi(\tilde{p}))}{\omega_p}\big(\underset{\overset{\shortparallel}{\varphi_*}\ \text{(former notation)}}{d\varphi_{\tilde{p}}}(v_1), \dots, d\varphi_{\tilde{p}}(v_n) \big)$$

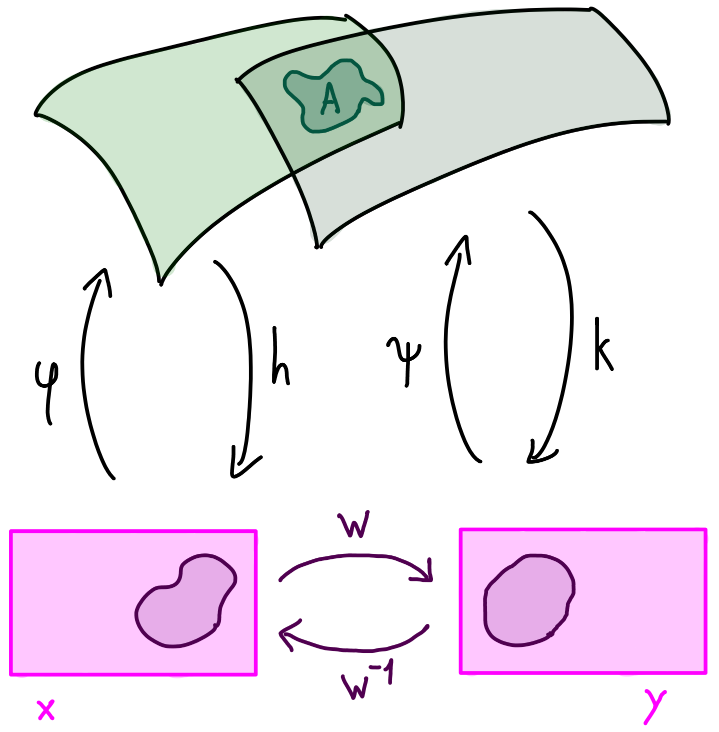
(2) $$(\varphi^*\omega)_{\tilde{p}} = f(\tilde{p}) \cdot \det(\cdot, \dots, \cdot) \qquad \text{(volume form on } \mathbb{R}^n\text{)}$$

$$f(\tilde{p}) = (\varphi^*\omega)_{\tilde{p}}(e_1, \dots, e_n) = \omega_p\big(\underset{\overset{\shortparallel}{\partial_1^{(h)}(p)}}{\varphi_*(e_1)}, \dots, \underset{\overset{\shortparallel}{\partial_n^{(h)}(p)}}{\varphi_*(e_n)}\big) = \omega_{1,2,\dots,n}(p)$$

(3)

$$\begin{array}{ccc} \displaystyle\int_{\tilde{U}} f(x)\, dx & \overset{\text{part 38}}{=} & \displaystyle\int_{h[U]} \varphi^*\omega \\ \| & & \| \\ \displaystyle\int_{h[U]} \omega_{1,2,\dots,n}(h^{-1}(x))\, dx & \overset{\text{part 39}}{=} & \displaystyle\int_U \omega \end{array}$$

Question: $\int_A \omega := \int_{h[A]} \varphi^* \omega$ well-defined?



transition map
diffeomorphism

$$\int_{h[A]} \varphi^*\omega \overset{?}{=} \int_{k[A]} \psi^*\omega$$

Proof: We have: $\psi \circ w = \varphi$ (restricted to a suitable subset)

$$\Longrightarrow \quad w^* \underbrace{\psi^*\omega}_{\tilde\omega} = \varphi^*\omega$$

$$\tilde\omega \rightsquigarrow \tilde\omega_y = g(y)\cdot \det(\cdot,\dots,\cdot)$$

$$\Longrightarrow \quad (w^*\tilde\omega)_x(v_1,\dots,v_n) = \tilde\omega_{w(x)}(dw_x(v_1),\dots,dw_x(v_n))$$

can be described by the Jacobian

$$= \tilde\omega_{w(x)}(J_w(x)v_1,\dots,J_w(x)v_n)$$

$$\overset{\text{part 35}}{=} \underbrace{\det(J_w(x))}_{>0}\cdot\tilde\omega_{w(x)}(v_1,\dots,v_n)$$

(everything should be orientation preserving)

Hence:

$$\int_{h[A]}\varphi^*\omega = \int_{h[A]} w^*\psi^*\omega = \int_{h[A]} \det(J_w(x))\, g(w(x))\,dx$$

ordinary integral in $\mathbb{R}^n$

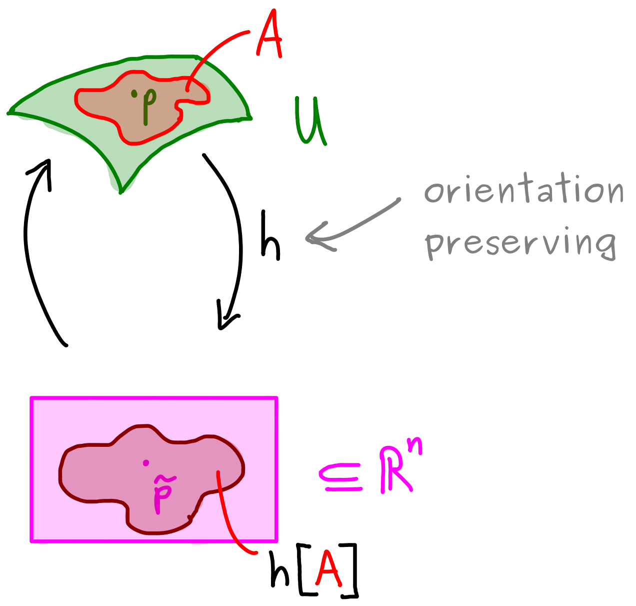
change of variables formula
$y = w(x)$

$$= \int_{k[A]} g(y)\,dy = \int_{k[A]}\psi^*\omega \qquad \square$$

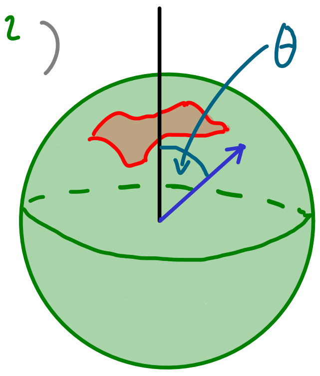
Manifolds – Part 41

We already know:

$$\int_A \omega := \int_{h[A]} \varphi^* \omega$$



Example: ω canonical volume form on S^2 (measures areas on S^2)



$$\Phi : \underbrace{(0,\pi)\times(0,2\pi)}_{= \tilde{U}} \longrightarrow \mathbb{R}^3$$

$$(\theta,\varphi) \longmapsto \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

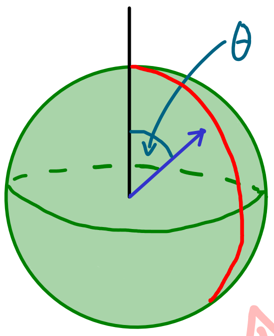
$$\int_{\Phi[\tilde{U}]} \omega = \int_{\tilde{U}} \Phi^* \omega$$

canonical volume form: $\omega(p) = \underbrace{\sqrt{\det(G(p))}}_{\sin(\theta)} \; \underset{\overset{\|}{d\theta}}{dx^1_p} \wedge \underset{\overset{\|}{d\varphi}}{dx^2_p}$

for $p = \Phi(\theta,\varphi)$ 1-forms on S^2

$$(\Phi^*\omega)(\underset{\overset{\|}{\binom{\theta}{\varphi}}}{\tilde{p}}) = \sin(\theta)\cdot \underbrace{\det(\cdot,\cdot)}_{d\theta \wedge d\varphi}$$

1-forms on $\tilde{U} \subseteq \mathbb{R}^2$



in short: $\omega = \sin(\theta)\ d\theta \wedge d\varphi$

$\Phi^{*}\omega = \sin(\theta)\ d\theta \wedge d\varphi$

$$\int_{S^2\setminus\{\dots\}} \omega = \int_{\Phi[\tilde{U}]} \omega = \int_{(0,\pi)\times(0,2\pi)} \Phi^{*}\omega = \int_{(0,\pi)\times(0,2\pi)} \sin(\theta)\ d\theta \wedge d\varphi$$

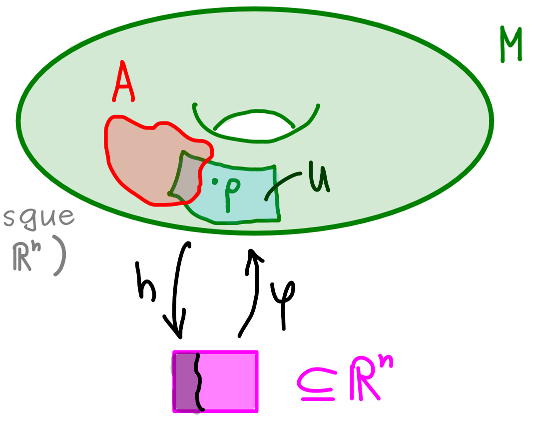
null set

$$= \int_0^{\pi}\left(\int_0^{2\pi} \sin(\theta)\ d\varphi\right) d\theta = 4\pi$$

Definition: Let M be an orientable n-dimensional manifold and $\omega \in \Omega^n(M)$.

A set $A \subseteq M$ is called

- **measurable** if $h[A \cap U]$ is measurable (w.r.t. Lebesgue measure in $\mathbb{R}^n$) for every chart (U,h).
- **null set** (set with measure zero) if $h[A \cap U]$ has Lebesgue measure 0 for every chart (U,h).

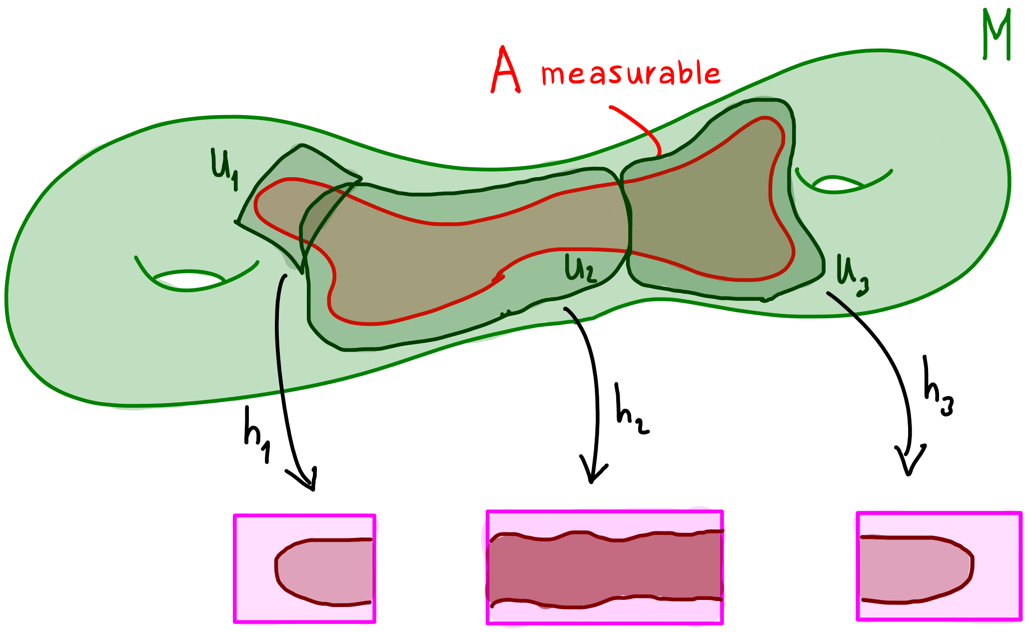


We get: $\int_A \omega$ is defined for every measurable set $A \subseteq U$ (where (U,h) is a chart) (assuming $\int_{h[A]} \varphi^{*}\omega$ exists in $\mathbb{R}^n$)

and $\int_{\mathcal{B}} \omega := \int_{\mathcal{B}\setminus\mathcal{N}} \omega$ if $\mathcal{B}\setminus\mathcal{N} \subseteq U$ (where (U,h) is a chart) and $\mathcal{N}$ is a null set.

Hence: $\int_{S^2} \omega = 4\pi$

Manifolds - Part 42



ω volume form

↳ put together to get: $\int_A \omega$

Fact: Every manifold M has a countable atlas $(U_k, h_k)_{k \in \mathbb{N}}$, which means

$$\bigcup_{k \in \mathbb{N}} U_k = M.$$

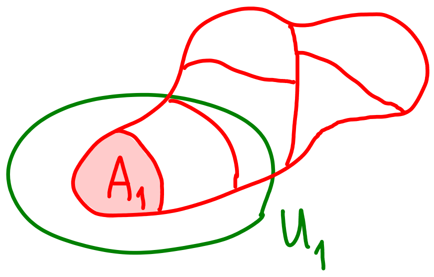
Lemma: Let M be an orientable n-dimensional manifold and $(U_k, h_k)_{k \in \mathbb{N}}$ atlas. Any measurable set $A \subseteq M$ can be decomposed into sets A_k:

(1) A_k is measurable for all $k \in \mathbb{N}$

(2) $\bigcup_{k \in \mathbb{N}} A_k = A$

(3) $A_i \cap A_j = \emptyset$ for $i \neq j$

(4) $A_k \subseteq U_k$ for all $k \in \mathbb{N}$



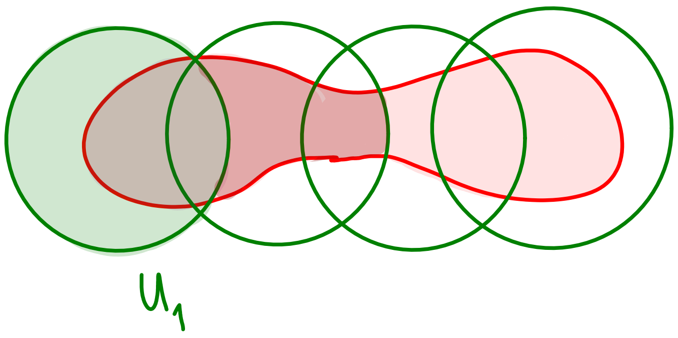
Proof: Just define:

$$A_1 := A \cap U_1$$

$$A_2 := (A \cap U_2) \setminus A_1$$

$$A_3 := (A \cap U_3) \setminus (A_1 \cup A_2)$$

$$\vdots$$

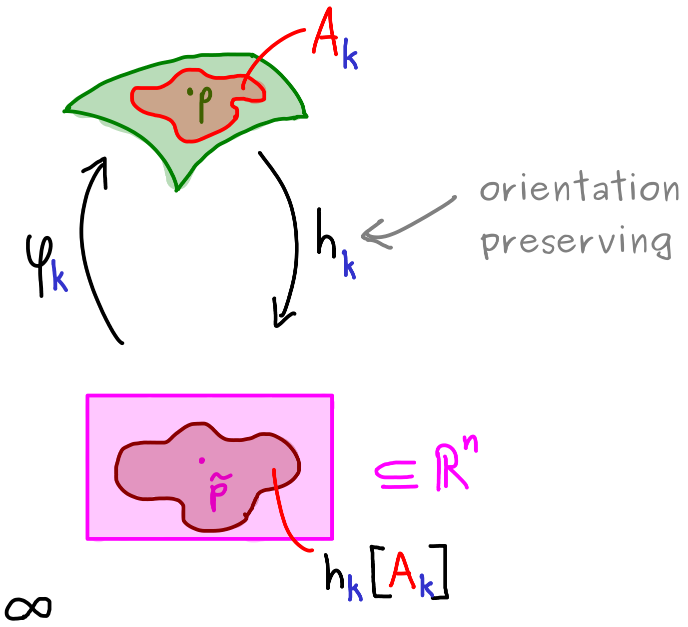


□

Definition: Let M be an orientable n-dimensional manifold and $\omega \in \Omega^n(M)$.

Choose A, A_k, (U_k, h_k) as in the Lemma before.

If (1) $\underbrace{\int_{A_k} \omega \text{ exists}}_{\int_{h_k[A_k]} \varphi_k^{*}\omega \text{ exists}}$ for all $k \in \mathbb{N}$



which means:

$$\int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x < \infty$$

component function: $\omega_{1,2,\dots,n}(p) = \omega_p(\partial_1, \partial_2, \dots, \partial_n)$

(2)

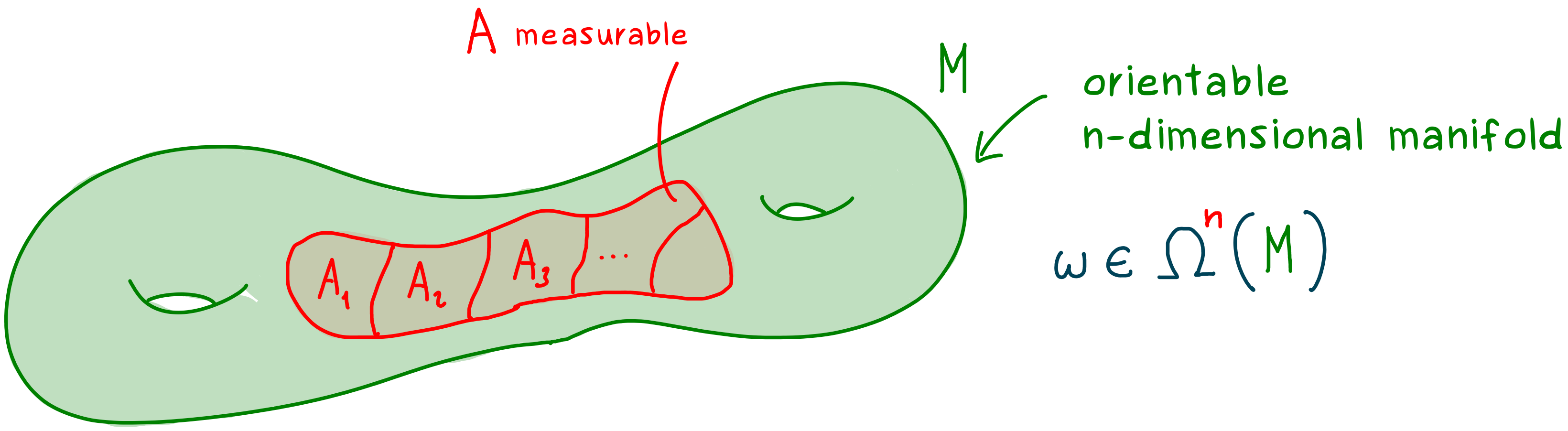
$$\sum_{k=1}^{\infty} \int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x < \infty,$$

then:

$$\int_A \omega := \sum_{k=1}^{\infty} \int_{A_k} \omega$$

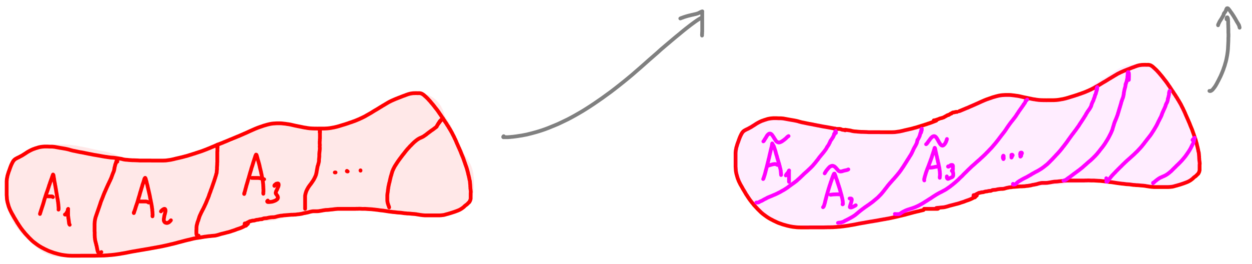
and if it works for $A = M$, then ω is called integrable.

Manifolds - Part 43



$\omega \in \Omega^n(M)$

$$\int_A \omega := \sum_{k=1}^{\infty} \int_{A_k} \omega \overset{\text{proof?}}{=} \sum_{m=1}^{\infty} \int_{\tilde{A}_m} \omega$$

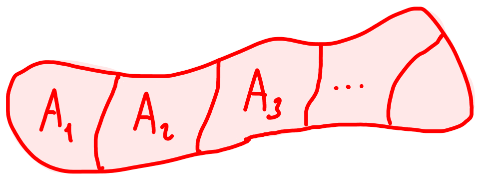


Proposition: (well-definedness of $\int_A \omega$)

$(U_k, h_k)_{k \in \mathbb{N}}$ atlas, $A = \bigcup_{k \in \mathbb{N}} A_k$ disjoint $A_k \subseteq U_k$ with:

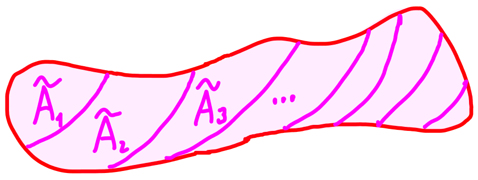
(1) $\int_{A_k} \omega$ exists for all $k \in \mathbb{N}$

(2)
$$\sum_{k=1}^{\infty} \int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}\left(h_k^{-1}(x) \right) \right| \, d^n x < \infty$$



$(\tilde{U}_m, \tilde{h}_m)_{m \in \mathbb{N}}$ atlas, $A = \bigcup_{m \in \mathbb{N}} \tilde{A}_m$ disjoint $\tilde{A}_m \subseteq \tilde{U}_m$.

(measurable)



Then:

$(\tilde{1})$ $\int_{\tilde{A}_m} \omega$ exists for all $m \in \mathbb{N}$

$(\tilde{2})$
$$\sum_{m=1}^{\infty} \int_{\tilde{h}_m[\tilde{A}_m]} \left| \omega_{1,2,\dots,n}\left(\tilde{h}_m^{-1}(x) \right) \right| \, d^n x < \infty$$

and:

$$\sum_{m=1}^{\infty} \int_{\tilde{h}_m[\tilde{A}_m]} \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \; d^n x = \sum_{k=1}^{\infty} \int_{h_k[A_k]} \omega_{1,2,\dots,n}(h_k^{-1}(x)) \; d^n x = \int_A \omega$$

Proof:

new decomposition: $A = \bigcup_{k,m} (A_k \cap \widehat{A}_m)$

$$\int_{h_k[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x \overset{\text{part 40}}{=} \int_{\tilde{h}_m[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \right| d^n x$$

$$\Rightarrow \sum_{m=1}^{\infty} \int_{h_k[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x = \sum_{m=1}^{\infty} \int_{\tilde{h}_m[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \right| d^n x$$

$$\parallel$$

$$\int_{\bigcup_{m \in \mathbb{N}} h_k[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x$$

$$\parallel \quad \int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x$$

finite!

$$\Rightarrow \sum_{k=1}^{\infty} \int_{h_k[A_k]} \left| \omega_{1,2,\dots,n}(h_k^{-1}(x)) \right| d^n x = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \int_{\tilde{h}_m[A_k \cap \widehat{A}_m]} \left| \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \right| d^n x$$

$$= \sum_{m=1}^{\infty} \int_{\tilde{h}_m[\widehat{A}_m]} \left| \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \right| d^n x$$

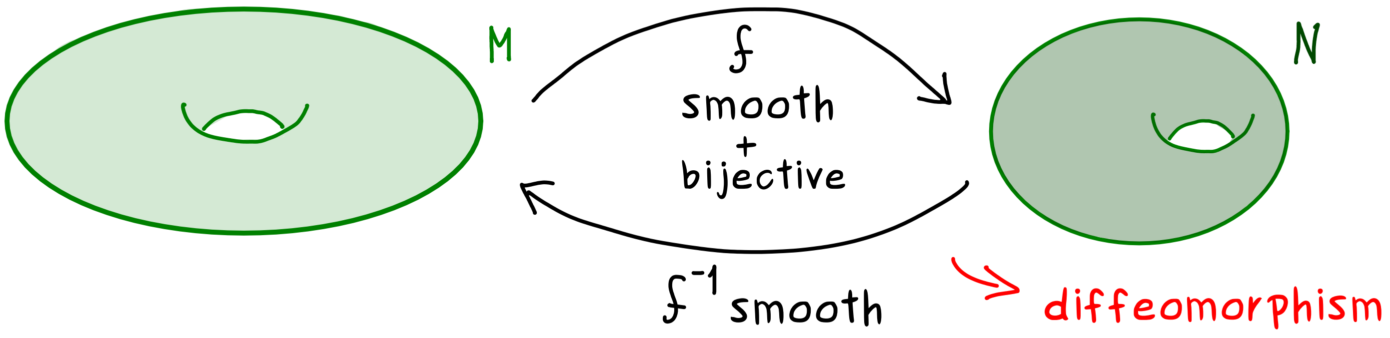
same calculation without absolute value

$$\Rightarrow \sum_{k=1}^{\infty} \int_{h_k[A_k]} \omega_{1,2,\dots,n}(h_k^{-1}(x)) \; d^n x = \sum_{m=1}^{\infty} \int_{\tilde{h}_m[\widehat{A}_m]} \omega_{1,2,\dots,n}(\tilde{h}_m^{-1}(x)) \; d^n x$$

$\square$

Manifolds - Part 44

Change of variables:



If $f: M \longrightarrow N$ is a diffeomorphism

and orientation preserving, then:

$$\int_M f^*\omega = \int_{f[M]} \omega$$

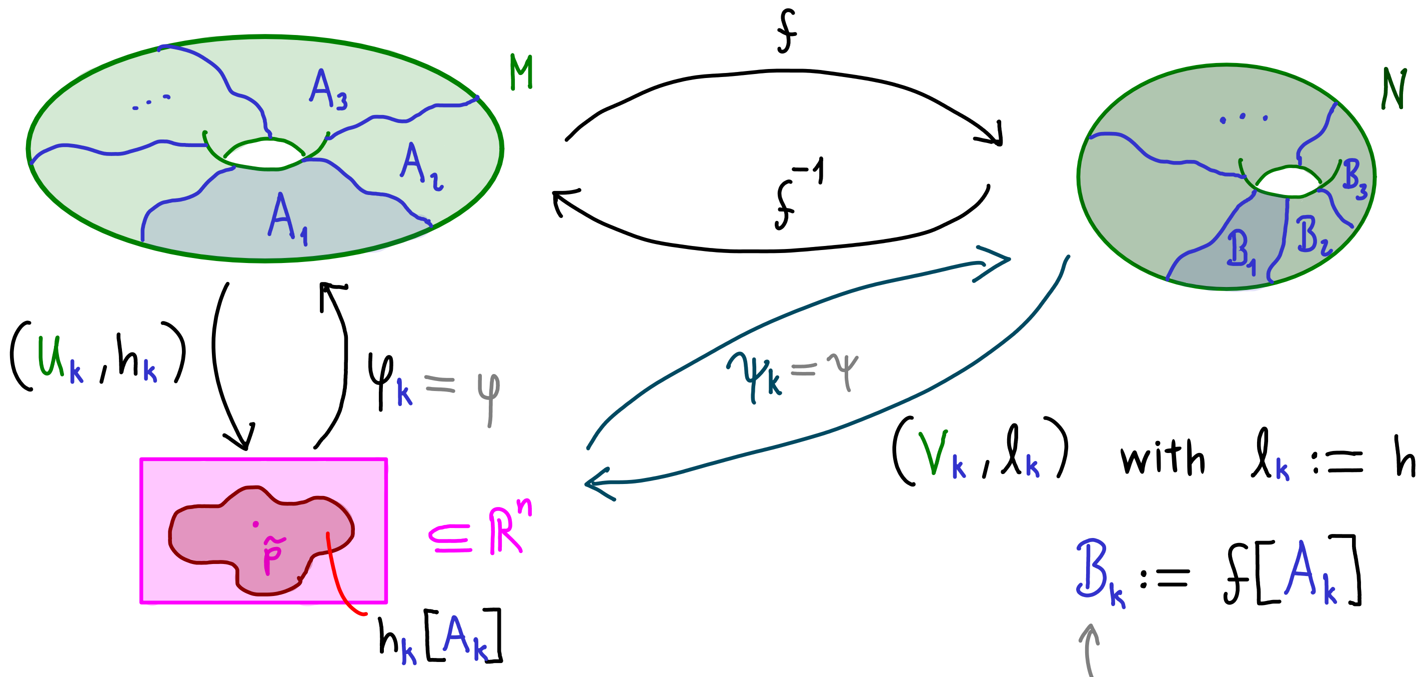
$(v_1, v_2, \dots, v_n)$ positively orientated in T_pM

$\Longrightarrow$ $(df_p(v_1), df_p(v_2), \dots, df_p(v_n))$

positively orientated in $T_{f(p)}N$

$$(f^*\omega)_p(v_1, v_2, \dots, v_n)$$
$$= \omega_{f(p)}(df_p(v_1), df_p(v_2), \dots, df_p(v_n))$$

Proof:



(V_k, ℓ_k) with $\ell_k := h_k \circ f^{-1}$

$B_k := f[A_k]$

countably many (pairwise) disjoint sets

decomposition of M into

countably many (pairwise) disjoint sets $A_k \subseteq U_k$

$$\int_{A_k} f^*\omega = \int_{h_k[A_k]} \varphi^* f^* \omega \qquad \text{with} \quad \psi = f \circ \varphi$$

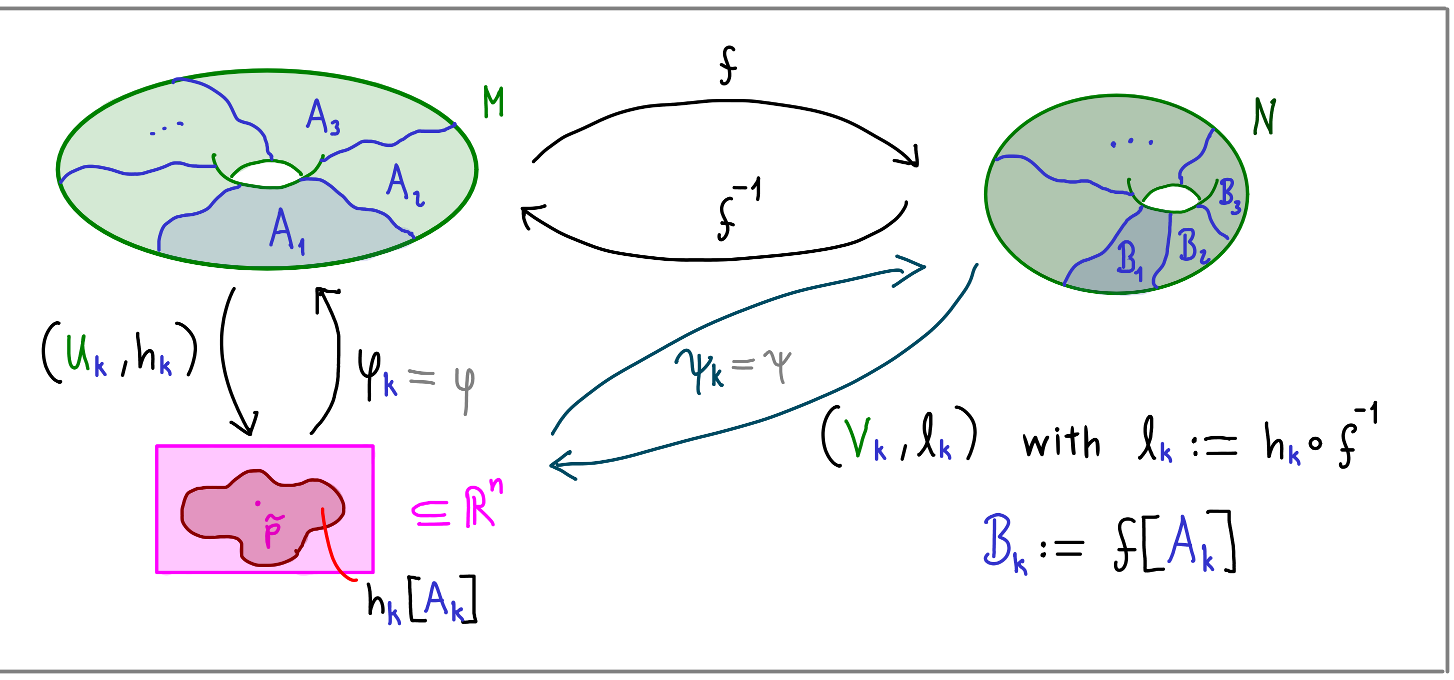
We have: $(\varphi^* f^* \omega)_x (u_1, u_2, \dots, u_n) = (f^*\omega)_{\varphi(x)} \big(d\varphi_x(u_1), \dots, d\varphi_x(u_n) \big)$

$$= \omega_{f(\varphi(x))} \big(df_{\varphi(x)}\, d\varphi_x(u_1), \dots, df_{\varphi(x)}\, d\varphi_x(u_n) \big)$$

$$= \omega_{\psi(x)} \big(d\psi_x(u_1), \dots, d\psi_x(u_n) \big)$$

$$= (\psi^* \omega)_x (u_1, u_2, \dots, u_n) \qquad \Longrightarrow (f \circ \varphi)^* = \varphi^* f^*$$

Result: $\displaystyle \int_{A_k} f^*\omega = \int_{h_k[A_k]} \varphi^* f^* \omega = \int_{h_k[A_k]} \psi^* \omega = \int_{\ell_k[B_k]} \psi^* \omega = \int_{B_k} \omega$



(V_k, ℓ_k) with $\ell_k := h_k \circ f^{-1}$

$B_k := f[A_k]$

$$\ell_k[B_k] = (h_k \circ f^{-1})[B_k] = h_k[A_k]$$

$$\overset{\sum_k}{\Longrightarrow} \int_M f^*\omega = \int_{f[M]} \omega$$

on both sides

$\square$