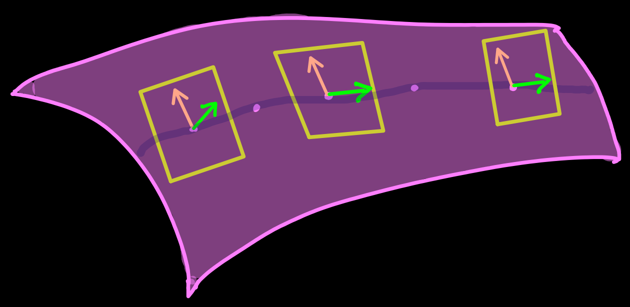
Manifolds – Part 32



orientable manifold M

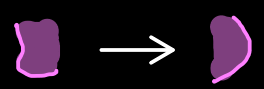
Fact: Let M be an n-dim smooth manifold. Then the following claims are equivalent:

(a) M is orientable : We have $\{(T_pM, or_p)\}$ such that

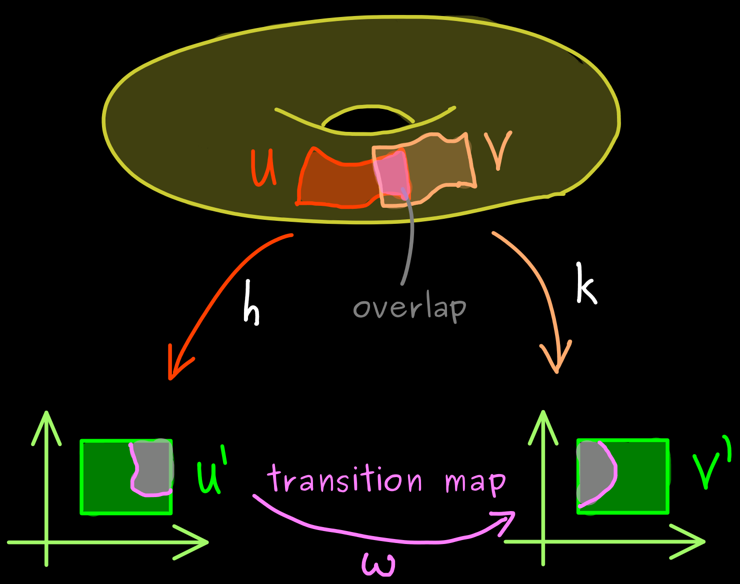
$$\forall p\in M \;\; \exists (U,h) \;\; \forall x\in U : \quad \left(\partial_1^{(h)}(x), \partial_2^{(h)}(x), \dots, \partial_n^{(h)}(x)\right) \in or_x$$

(b) There is an atlas for M — collection of charts that cover the manifold

such that all transition maps

$\omega :$  satisfy:

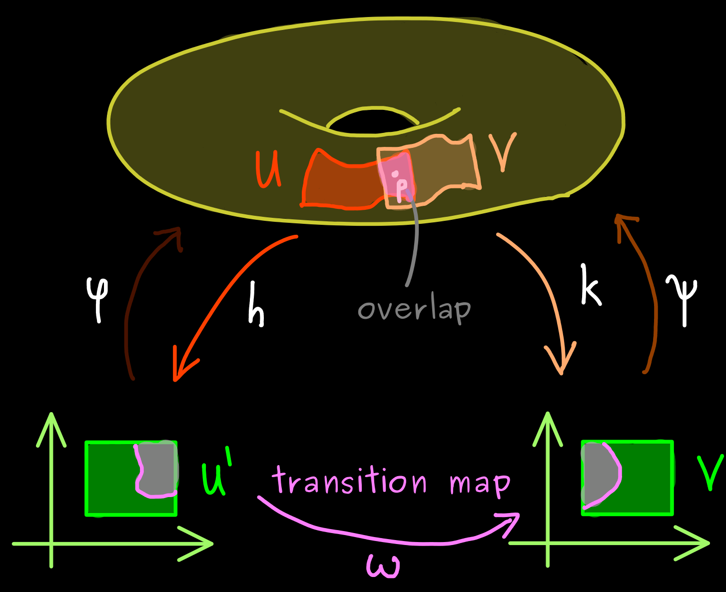
$$\det(J_\omega(x)) > 0$$

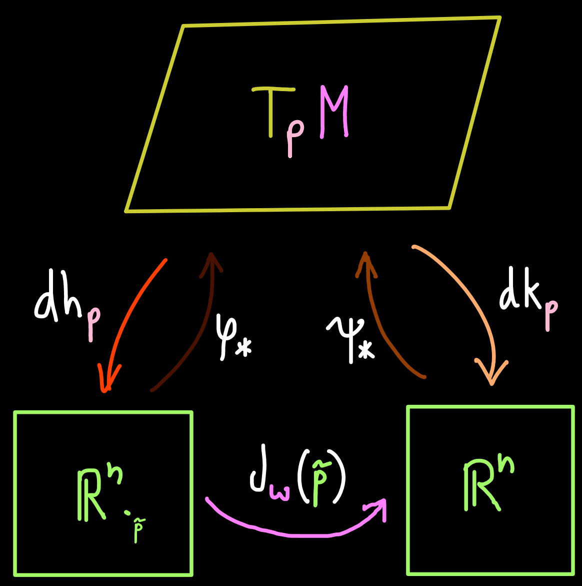


(c) There is a differential form (volume form)

$\omega \in \Omega^n(M)$ with $\omega(p) \neq 0$ for all $p \in M$.

Proof: (a) $\iff$ (b)





We have: $\psi_*\big(\underbrace{J_\omega(\tilde{p})\, e_1}_{\text{first column of Jacobian}}\big) = \varphi_*(e_1)$

$$= \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \sum_j \lambda_j e_j$$

$$\implies \sum_{j=1}^n \lambda_j \underbrace{\psi_*(e_j)}_{\partial_j^{(k)}(p)} = \underbrace{\varphi_*(e_1)}_{\partial_1^{(h)}(p)} \quad (*)$$

Change-of-basis matrix: $\mathcal{B} = \left(\partial_1^{(h)}(p), \dots, \partial_n^{(h)}(p)\right) \overset{T_{\mathcal{C}\leftarrow\mathcal{B}}}{\rightsquigarrow} \mathcal{C} = \left(\partial_1^{(k)}(p), \dots, \partial_n^{(k)}(p)\right)$

$$\overset{(*)}{\implies} T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \lambda_1 & \cdots \\ \lambda_2 & \\ \vdots & \\ \lambda_n & \end{pmatrix} = J_\omega(\tilde{p})$$

Hence:

$$\det(T_{\mathcal{C}\leftarrow\mathcal{B}}) > 0 \iff \det(J_\omega(x)) > 0$$

$$\text{(a)} \iff \text{(b)}$$