BECOME A MEMBER

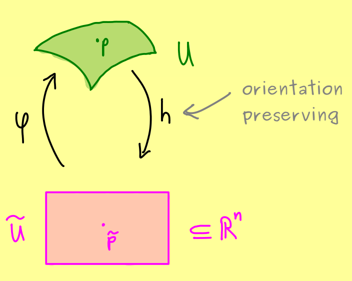
ON STEADY

The Bright Side of
Mathematics

Manifolds – Part 40

Let M be an orientable n-dimensional manifold and $\omega \in \Omega^n(M)$.

$$\omega(p) = \underbrace{\omega_{1,2,\dots,n}(p)}_{\text{component function}} \, dx_p^1 \wedge dx_p^2 \wedge \dots \wedge dx_p^n$$



$$\int_U \omega := \int_{h[U]} \omega_{1,2,\dots,n}(h^{-1}(x)) \, dx \quad \text{(integral in } \mathbb{R}^n\text{)}$$

$$= \int_{h[U]} \varphi^* \omega$$

← volume form on manifold $\mathbb{R}^n$

Some explanations: (1) For $\omega \in \Omega^n(U)$, $\varphi : \tilde{U} \to U$, we define $\varphi^* \omega \in \Omega^n(\tilde{U})$

by: $(\varphi^*\omega)_{\tilde{p}}(v_1, \dots, v_n) := \omega_p\big(d\varphi_{\tilde{p}}(v_1), \dots, d\varphi_{\tilde{p}}(v_n)\big)$

$(p = \varphi(\tilde{p}))$ $\quad d\varphi_{\tilde{p}} = \varphi_*$ (former notation)

(2) $(\varphi^*\omega)_{\tilde{p}} = f(\tilde{p}) \cdot \det(\cdot, \dots, \cdot)$ (volume form on $\mathbb{R}^n$)

$$f(\tilde{p}) = (\varphi^*\omega)_{\tilde{p}}(e_1, \dots, e_n) = \omega_p\big(\varphi_*(e_1), \dots, \varphi_*(e_n)\big) = \omega_{1,2,\dots,n}(p)$$

with $\varphi_*(e_1) = \partial_1^{(h)}(p)$, …, $\varphi_*(e_n) = \partial_n^{(h)}(p)$

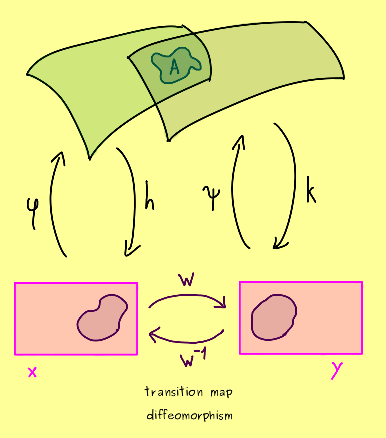
(3)

$$\int_{\tilde{U}} f(x) \, dx \overset{\text{part 38}}{=} \int_{h[U]} \varphi^* \omega$$

$$\| \qquad\qquad \|$$

$$\int_{h[U]} \omega_{1,2,\dots,n}(h^{-1}(x)) \, dx \overset{\text{part 39}}{=} \int_U \omega$$

Question: $\displaystyle\int_A \omega := \int_{h[A]} \varphi^* \omega$ well-defined?



$$\int_{h[A]} \varphi^* \omega \overset{?}{=} \int_{k[A]} \psi^* \omega$$

Proof: We have: $\psi \circ w = \varphi$ (restricted to a suitable subset)

$$\Rightarrow \quad w^* \underbrace{\psi^* \omega}_{\tilde{\omega}} = \varphi^* \omega$$

$$\tilde{\omega} \rightsquigarrow \tilde{\omega}_y = g(y) \cdot \det(\cdot, \dots, \cdot)$$

$$\Rightarrow \quad (w^* \tilde{\omega})_x (v_1, \dots, v_n) = \tilde{\omega}_{w(x)}\big(dw_x(v_1), \dots, dw_x(v_n)\big)$$

← can be described by the Jacobian

$$= \tilde{\omega}_{w(x)}\big(J_w(x) v_1, \dots, J_w(x) v_n\big)$$

$$\overset{\text{part 35}}{=} \underbrace{\det(J_w(x))}_{>0} \cdot \tilde{\omega}_{w(x)}(v_1, \dots, v_n)$$

(everything should be orientation preserving)

Hence:

$$\int_{h[A]} \varphi^* \omega = \int_{h[A]} w^* \psi^* \omega = \int_{h[A]} \det(J_w(x)) \, g(w(x)) \, dx$$

ordinary integral in $\mathbb{R}^n$

change of variables formula $y = w(x)$ →

$$= \int_{k[A]} g(y) \, dy = \int_{k[A]} \psi^* \omega \qquad \square$$