The Bright Side of Mathematics

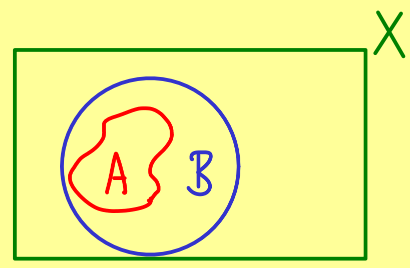
Outer measures – part 1

- tools for the proof of Carathéodory's extension theorem
- "outer measure" is a new notion

  not an attribute for "measure"!

  ↳ outer measures don't have to be measures!

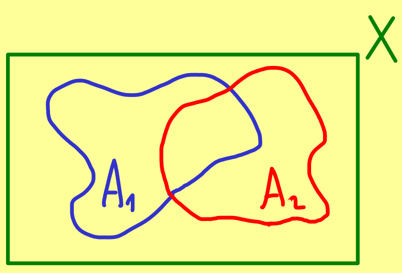
Definition: A map $\varphi: \underbrace{P(X)}_{\text{power set}} \longrightarrow [0,\infty]$ is called an outer measure if:



(a) $\varphi(\emptyset) = 0$

(b) $A \subseteq B \implies \varphi(A) \leq \varphi(B)$ (monotonicity)

(c) $A_1, A_2, \ldots \in P(X) \implies$



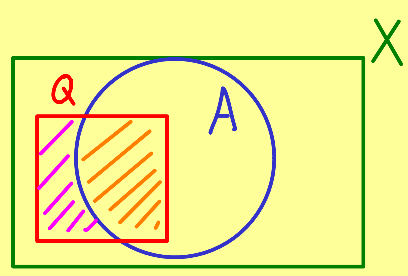
$$\varphi\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \varphi(A_n)$$

(σ-subadditivity)

Question: $\varphi: P(X) \longrightarrow [0,\infty]$ äußeres Maß $\overset{?}{\rightsquigarrow}$ μ measure ?

outer measure

Definition: Let φ be an outer measure. $A \in P(X)$ is called φ-measurable if for all $Q \in P(X)$ we have:



$$\varphi(Q) \geq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

Important proposition:

If $\varphi: P(X) \longrightarrow [0,\infty]$ is an outer measure, then:

- $\mathcal{A}_\varphi := \{A \subseteq X \mid A \ \varphi\text{-measurable}\}$ is a σ-algebra
- $\mu: \mathcal{A}_\varphi \longrightarrow [0,\infty]$, $\mu(A) = \varphi(A)$, is a measure