The Bright Side of Mathematics

Ordinary Differential Equations – Part 2

Definitions: For $I \subseteq \mathbb{R}$ (interval, open set, union intervals,...)

$$C^k(I) := \left\{ \begin{array}{l} x: I \longrightarrow \mathbb{R} \\ \quad t \longmapsto x(t) \end{array} \;\middle|\; \underbrace{x \text{ is } k\text{-times continuously differentiable}}_{\dot{x},\, \ddot{x},\, \dots,\, x^{(k)} \text{ continuous functions}} \right\}$$

$$\dot{x} = x' = \frac{dx}{dt}$$

Ordinary differential equation: $F(t, x, \dot{x}, \dots, x^{(k)}) = 0$

$\boxed{\text{ODE}}$ continuous

Example: $t + x + 2\dot{x} + (\ddot{x})^2 = 0$

(explicit) ODE of order 1: $\dot{x} = w(t, x)$, $w: I \times J \longrightarrow \mathbb{R}$, $I, J \subseteq \mathbb{R}$ intervals

Example: $\dot{x} = x + t$

What about? $\begin{pmatrix} \dot{x}_1 = x_2 + t \\ \dot{x}_2 = x_1 + t \end{pmatrix} \rightsquigarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = w\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

System of (explicit) ODEs of order 1:

$$\dot{x} = w(t, x), \quad x(t) \in \mathbb{R}^n, \quad w: I \times U \longrightarrow \mathbb{R}^n$$

open set in $\mathbb{R}^n$

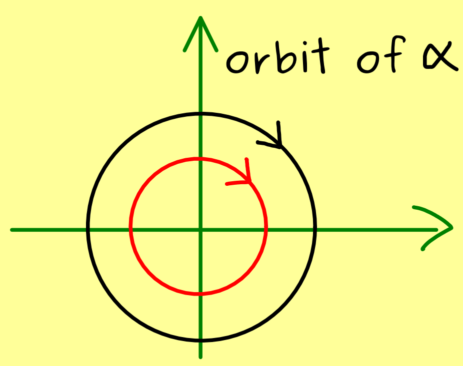
Solution of ODE: $\alpha: (t_0, t_1) \longrightarrow U$ with $(t_0, t_1) \subseteq I$

satisfies $\dot{\alpha}(t) = w(t, \alpha(t))$ for all $t \in (t_0, t_1)$.

Example: $\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{array}$, $n = 2$, $U = \mathbb{R}^2$, $w(t, x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$

$\rightsquigarrow \alpha(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ is a solution

$\tilde{\alpha}(t) = \frac{1}{2} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ is a solution



orbit of α

orbit of $\tilde{\alpha}$