BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Ordinary Differential Equations – Part 6

non-autonomous ODE: $\dot{x} = w(t,x)$ can we separate t and x?

example: $\dot{x} = \underbrace{t^3}_{\text{only } t} \cdot \underbrace{x^2}_{\text{only } x}$

Separation of variables: $\dot{x} = g(t)\cdot h(x)$, $x(t_0) = x_0$ (initial value problem)

continuous functions

Assume: $h(x_0) \neq 0 \quad \Rightarrow \quad \dfrac{\dot{x}}{h(x)} = g(t)$

Therefore: any solution $\alpha : \underset{\ni t_0}{(t_1,t_2)} \longrightarrow \mathbb{R}$ with $\alpha(t_0) = x_0$ satisfies:

fundamental theorem of calculus

$$\frac{\dot{\alpha}(s)}{h(\alpha(s))} = g(s) \quad \text{for all } s \in (t_1,t_2)$$

$$\Longleftrightarrow \int_{t_0}^{t} \frac{\dot{\alpha}(s)}{h(\alpha(s))}\, ds = \int_{t_0}^{t} g(s)\, ds \quad \text{for all } t \in (t_1,t_2)$$

substitution: $x = \alpha(s)$, $dx = \dot{\alpha}(s)\, ds$

$$\Longleftrightarrow \int_{x_0}^{\alpha(t)} \frac{1}{h(x)}\, dx = \int_{t_0}^{t} g(s)\, ds \quad \text{for all } t \in (t_1,t_2)$$

$$\Longleftrightarrow F(\alpha(t)) - F(x_0) = G(t) - G(t_0) \quad \text{for all } t \in (t_1,t_2)$$

where F is an antiderivative of $\frac{1}{h}$

where G is an antiderivative of g

$$\Longleftrightarrow F(\alpha(t)) = G(t) + c$$

for a constant $c \in \mathbb{R}$, for all $t \in (t_1,t_2)$

$$\Longleftrightarrow \alpha(t) = F^{-1}\big(G(t) + c\big)$$

Example: (a) $\dot{x} = \frac{1}{3}t^3 x$, $x(0) = x_0 \neq 0$

$$\Longleftrightarrow \frac{dx}{dt} = \frac{1}{3}t^3 x \quad \overset{\text{informally}}{\Longleftrightarrow} \quad \int \frac{dx}{x} = \int \frac{1}{3}t^3\, dt$$

$$\Longleftrightarrow \log(|x|) = \frac{1}{12}t^4 + c \qquad \text{for a constant } c \in \mathbb{R}$$

natural logarithm

$$\Longleftrightarrow |\alpha(t)| = e^{\frac{1}{12}t^4 + c} \quad \overset{\alpha(0) = x_0}{\Longrightarrow} \quad \alpha(t) = x_0 \cdot e^{\frac{1}{12}t^4}$$

(b) $\dot{x} = \sin(t)\cdot e^x$, $x(0) = x_0$

$$\Longleftrightarrow \frac{dx}{dt} = \sin(t)\cdot e^x \quad \overset{\text{informally}}{\Longleftrightarrow} \quad \int \frac{dx}{e^x} = \int \sin(t)\, dt$$

$$\Longleftrightarrow -e^{-x} = -\cos(t) + c \qquad \text{for a constant } c \in \mathbb{R}$$

$$\Longleftrightarrow \alpha(t) = -\log(\cos(t) + \tilde{c}) \qquad \text{for a constant } \tilde{c} \in \mathbb{R}$$

$$\overset{\alpha(0) = x_0}{\Longrightarrow} -\log(\cos(0) + \tilde{c}) = x_0 \quad \Longrightarrow \quad \tilde{c} = e^{-x_0} - 1$$