The Bright Side of Mathematics

Ordinary Differential Equations – Part 14

initial value problem: $(\text{IVP}^{t_0}_{x_0})$ $\boxed{\begin{aligned} \dot{x} &= v(x) \\ x(t_0) &= x_0 \end{aligned}}$ with $v: \mathcal{D} \to \mathbb{R}^n$ loc. Lipschitz continuous

$\mathcal{D}$ open in $\mathbb{R}^n$

(Picard-Lindelöf theorem)

$\Longrightarrow$ there is $\varepsilon > 0$ and a unique solution

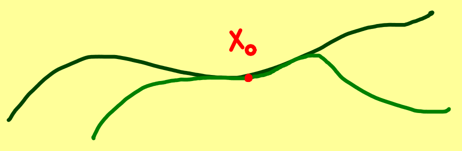
$$\alpha: (t_0-\varepsilon, t_0+\varepsilon) \longrightarrow \mathcal{D}$$

Extension of solution: We say a solution $\tilde{\alpha}: I \longrightarrow \mathcal{D}$ extends $\alpha: (t_0-\varepsilon, t_0+\varepsilon) \longrightarrow \mathcal{D}$ if $I \supsetneq (t_0-\varepsilon, t_0+\varepsilon)$ and $\tilde{\alpha}|_{(t_0-\varepsilon, t_0+\varepsilon)} = \alpha$.

Maximal solutions: A solution $\alpha: I \longrightarrow \mathcal{D}$ is called maximal if there is no extension.

Proposition: $(\text{IVP}^{t_0}_{x_0})$ for $v: \mathcal{D} \to \mathbb{R}^n$ loc. Lipschitz continuous has exactly one maximal solution (defined on an open interval).

Proof:



$\alpha_1: I_1 \longrightarrow \mathcal{D}$

$\alpha_2: I_2 \longrightarrow \mathcal{D}$ two solutions of $(\text{IVP}^{t_0}_{x_0})$

$\Longrightarrow I := I_1 \cap I_2 = (a, b)$

$\Longrightarrow \alpha_1|_I, \alpha_2|_I$ two solutions of $(\text{IVP}^{t_0}_{x_0})$

There is $\varepsilon > 0$ such that $\alpha_1|_{(t_0-\varepsilon, t_0+\varepsilon)} = \alpha_2|_{(t_0-\varepsilon, t_0+\varepsilon)}$

$$\left\{ J \text{ open interval} \;\middle|\; I \supseteq J \supseteq (t_0-\varepsilon, t_0+\varepsilon) \text{ with } \alpha_1|_J = \alpha_2|_J \right\} = \mathcal{M}$$

$(t_-, t_+) := \bigcup\limits_{J \in \mathcal{M}} J$ gives maximal open interval

Show: $t_+ = b$ Assume: $t_+ \neq b$

Then: $\alpha_1(t) = \alpha_2(t)$ for all $t \in (t_-, t_+)$

$\downarrow \; t \to t_+ \; \downarrow$

$\tilde{x}_0 = \alpha_1(t_+) = \alpha_2(t_+)$ because of continuity on I

Look at $(\text{IVP}^{t_+}_{\tilde{x}_0})$: uniqueness of solution implies:

$\alpha_1(t) = \alpha_2(t)$ for $t \in (t_+ - \tilde{\varepsilon}, t_+ + \tilde{\varepsilon})$

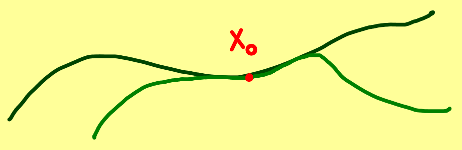
$\Longrightarrow (t_-, t_+ + \tilde{\varepsilon}) \in \mathcal{M}$ ↯

Conclusion: $(t_-, t_+) = I$ and

$\alpha_1|_I = \alpha_2|_I \Longrightarrow \alpha: I_1 \cup I_2 \longrightarrow \mathcal{D}$

Define: $\left\{ I \text{ open interval} \;\middle|\; \text{there is a solution } \alpha: I \longrightarrow \mathcal{D} \text{ for } (\text{IVP}^{t_0}_{x_0}) \right\} = \mathcal{S}$

$\bigcup\limits_{I \in \mathcal{S}} I$ open interval for maximal solution □



cannot happen!

Definition: If the maximal solution is defined on $I = \mathbb{R}$, then it's called a global solution.