BECOME A MEMBER

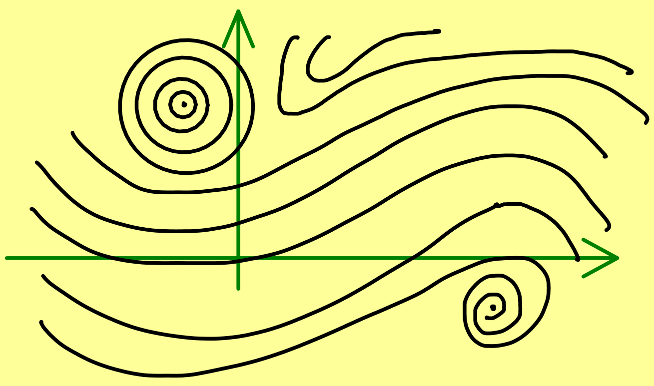
ON STEADY

The Bright Side of
Mathematics

Ordinary Differential Equations – Part 16

$\dot{x} = v(x)$, $\quad v: \mathcal{D} \to \mathbb{R}^n$

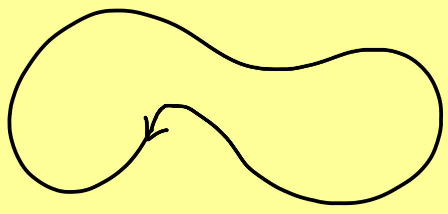
$\mathcal{D}$ open in $\mathbb{R}^n$



Definition: A global solution $\alpha: \mathbb{R} \to \mathcal{D}$ of $\dot{x} = v(x)$ is called:

- fixed point if $\alpha(t) = \alpha(0)$ for all $t \in \mathbb{R}$.
- periodic if there is a $T > 0$ with $\alpha(t+T) = \alpha(t)$ for all $t \in \mathbb{R}$.

T: a period



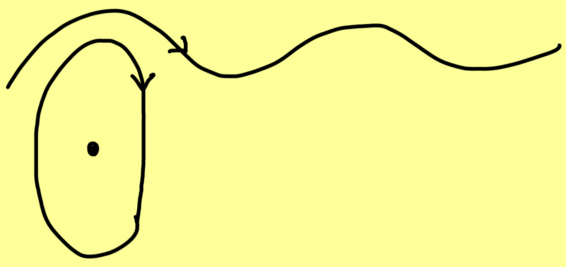
Proposition: For $v: \mathcal{D} \to \mathbb{R}^n$ loc. Lipschitz continuous,

there are three options for the maximal solution α of $\boxed{\begin{array}{l}\dot{x} = v(x)\\ x(0) = x_0\end{array}}$:

(a) α is injective

(b) α is fixed point

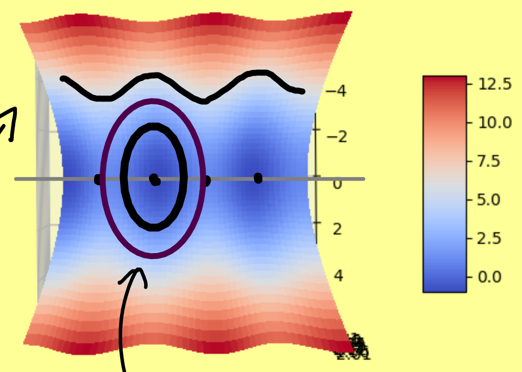
(c) α is periodic



Example: $\ddot{x} = -\sin(x) \rightsquigarrow \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ -\sin(x_1) \end{pmatrix} = v(x_1, x_2)$

Do we have $f: \mathbb{R}^2 \to \mathbb{R}$ with $f(\alpha(t)) = \text{constant}$ for all t?

contour lines of f



injective solution

periodic solution

Note: $f(\alpha(t)) = \text{constant}$ for all t

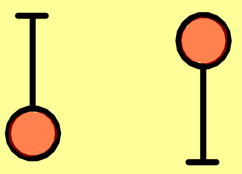
$\Longleftrightarrow \dfrac{d}{dt} f(\alpha(t)) = 0$ for all t

$\overset{\text{chain rule}}{\Longleftrightarrow} \langle \text{grad} f(\alpha(t)), \underbrace{\dot{\alpha}(t)}_{v(\alpha(t))} \rangle = 0$ for all t

$f(x_1, x_2) = \frac{1}{2} x_2^2 - \cos(x_1)$ satisfies $\langle \text{grad} f(x_1, x_2), v(x_1, x_2) \rangle = 0$.

Fixed point: $\text{grad} f(x_1, x_2) = \begin{pmatrix} \sin(x_1) \\ x_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow x_2 = 0, \; x_1 = k \cdot \pi$

$k \in \mathbb{Z}$



(pendulum does not move)