Ordinary Differential Equations - Part 18

Definition: A system of ODEs $\dot{x} = w(t,x)$ is called a <u>system of linear differential equations</u> if

$$w : \underbrace{\mathbb{R}}_{\text{or interval}} \times \underbrace{\mathbb{R}^n}_{\text{or open set}} \longrightarrow \mathbb{R}^n \,, \quad (t,x) \longmapsto A(t)\,x + b(t)$$

continuous

with $A : t \mapsto A(t) \in \mathbb{R}^{n\times n}$, $b : t \mapsto b(t) \in \mathbb{R}^n$ continuous

Note:
- If $b(t) = 0$ for all t, then the system is called <u>homogeneous</u>.
- If $A(t) = A$, $b(t) = b$ for all t, then the system is called <u>autonomous</u>.

Lipschitz condition?

$$\| w(t,x) - w(t,y) \| = \| A(t)x + b(t) - (A(t)y + b(t)) \|$$

$$= \| A(t)(x-y) \| \leq \underbrace{\| A(t) \|}_{\text{matrix norm/ operator norm}} \cdot \| x-y \|$$

$[-T,T] \ni t \mapsto \| A(t) \|$ continuous

$$\leq L_T \cdot \| x-y \|$$

Picard-Lindelöf theorem (special version)

$\Longrightarrow$ unique global solution $\alpha : \underbrace{\mathbb{R}}_{\text{or interval}} \to \mathbb{R}^n$ for initial value problem $\boxed{\begin{array}{l} \dot{x} = w(t,x) \\ x(t_0) = x_0 \end{array}}$.

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} - \begin{pmatrix} 1 \cdot x_1 \\ 2t \cdot x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix} \begin{array}{l} \leftarrow e^{-t} \\ \leftarrow e^{-t^2} \end{array}$$

$$\rightsquigarrow \begin{cases} \dot{x}_1 e^{-t} - x_1 e^{-t} = 0 \\ \dot{x}_2 e^{-t^2} - x_2 \cdot 2t\, e^{-t^2} = 1 \end{cases} \rightsquigarrow \begin{cases} \frac{d}{dt}\left(x_1 e^{-t}\right) = 0 \\ \frac{d}{dt}\left(x_2 e^{-t^2}\right) = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 e^{-t} = C_1 \\ x_2 e^{-t^2} = t + C_2 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} C_1 e^{t} \\ (t + C_2)\, e^{t^2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & 0 \\ 0 & e^{t^2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t e^{t^2} \end{pmatrix}$$