Ordinary Differential Equations – Part 19

System of linear differential equations: (of first order)

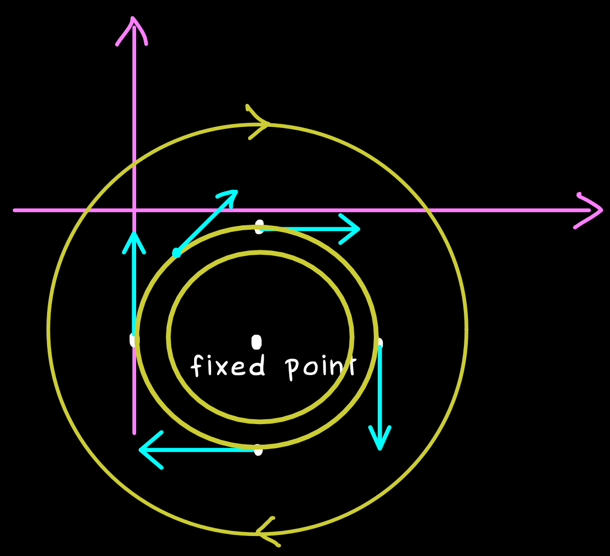
$$\dot{x} = A(t)\,x + b(t) \quad \text{with} \quad I \ni t \xrightarrow{\text{continuous}} A(t) \in \mathbb{R}^{n\times n}, \quad I \ni t \longmapsto b(t) \in \mathbb{R}^n$$

interval in $\mathbb{R}$

- solutions are global $\alpha : I \to \mathbb{R}^n$
- autonomous systems: $A(t) = A$, $b(t) = b$

  example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  $$v(x) = Ax + b$$



- corresponding homogeneous system

  $$\dot{x} = A(t)\,x$$

Fact: If $\alpha : I \to \mathbb{R}^n$, $\beta : I \to \mathbb{R}^n$ are two solutions of $\dot{x} = A(t)\,x$,

$$(\alpha + \beta)^{\boldsymbol{\cdot}}(t) = \dot{\alpha}(t) + \dot{\beta}(t) = A(t)\,\alpha(t) + A(t)\,\beta(t)$$

$$= A(t)\big(\alpha(t) + \beta(t)\big)$$

$$(\lambda \cdot \alpha)^{\boldsymbol{\cdot}}(t) = A(t)\big(\lambda \cdot \alpha(t)\big) \quad \rightsquigarrow \quad \boxed{\text{linear combinations of solutions are solutions}}$$

Proposition: The solution set of the corresponding homogeneous system

$$S_0 := \left\{ \alpha : I \to \mathbb{R}^n \ \text{continuously differentiable} \ \middle| \ \dot{\alpha}(t) = A(t)\,\alpha(t) \ \text{for all } t \in I \right\}$$

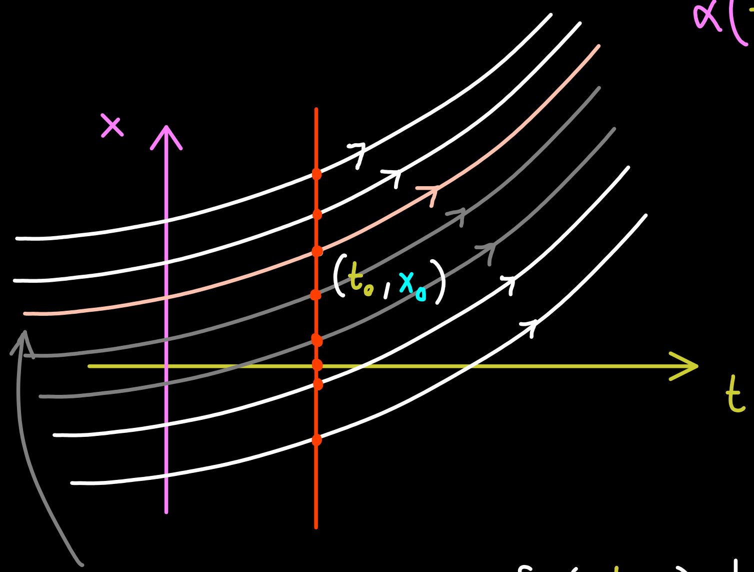
forms an n-dimensional $\mathbb{R}$-vector space.

Proof: S_0 is a subspace in the $\mathbb{R}$-vector space $C^1(I, \mathbb{R}^n)$.

What about the dimension of S_0 ?

$$(\mathrm{IVP}^{t_0}_{x_0}) \ \boxed{\begin{array}{l} \dot{x} = A(t)\,x \\ x(t_0) = x_0 \end{array}} \ \xrightarrow[\text{(special version)}]{\text{Picard-Lindelöf theorem}} \ \text{unique solution } \alpha : I \to \mathbb{R}^n, \quad \alpha(t_0) = x_0$$

extended phase portrait:



extended orbit at (t_0, x_0) : $\left\{ \begin{pmatrix} t \\ \alpha(t) \end{pmatrix} \ \middle| \ t \in I \text{ where } \alpha \text{ is unique solution of } (\mathrm{IVP}^{t_0}_{x_0}) \right\}$

define a map: $\ell : S_0 \longrightarrow \mathbb{R}^n$, $\quad \alpha \longmapsto \alpha(t_0)$ $\quad$ linear map!

$\hookrightarrow$ surjective (every $(\mathrm{IVP}^{t_0}_{x_0})$ has a solution)

$\hookrightarrow$ injective $\left(\ell(\alpha) = \ell(\beta) \Rightarrow \alpha(t_0) = \beta(t_0) \overset{\text{uniqueness}}{\Longrightarrow} \alpha = \beta \text{ on } I \right)$

$$\Longrightarrow \ \ell : S_0 \longrightarrow \mathbb{R}^n \ \text{isomorphism}$$

$$\Longrightarrow \ \dim(S_0) = \dim(\mathbb{R}^n) = n \qquad \square$$