The Bright Side of Mathematics

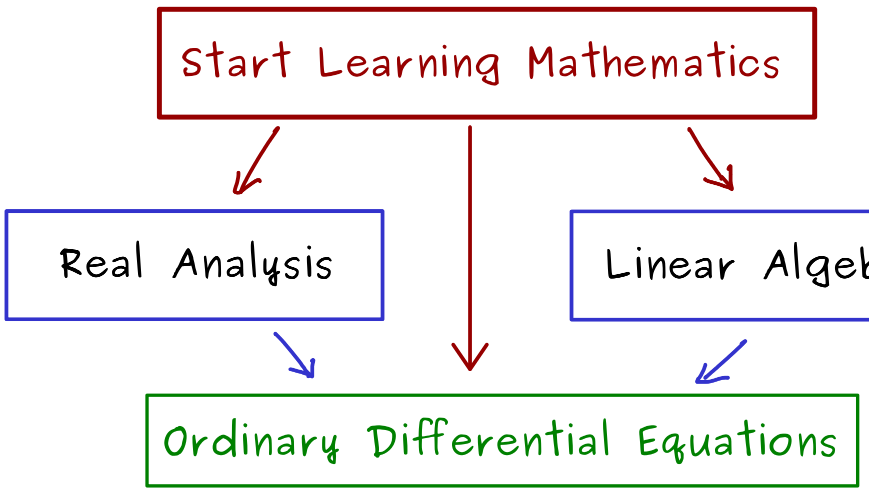
The following pages cover the whole Ordinary Differential Equations course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Ordinary Differential Equations – Part 1

$f' = f \longleftarrow$ search for a function f that satisfies this?

$f(x) = e^x$



Start Learning Mathematics

Real Analysis

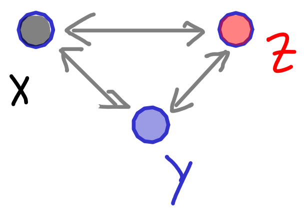
Linear Algebra

Ordinary Differential Equations

<u>Other examples:</u> (a) $\ddot{x} = -\omega^2 x$ (harmonic oscillator)

(second order derivatives)

(b)



$$\left.\begin{aligned} m\cdot\ddot{x} &= F \\ m\cdot\ddot{y} &= F \\ m\cdot\ddot{z} &= F \end{aligned}\right\} \text{system of differential equations}$$

Topics:

- system of ordinary differential equations (ODE)
- solution methods
- existence and uniqueness of solutions
- linear ordinary differential equations (matrix exponential function)

Ordinary Differential Equations – Part 2

Definitions: For $I \subseteq \mathbb{R}$ (interval, open set, union intervals,...)

$$C^k(I) := \left\{ \begin{array}{l} x: I \longrightarrow \mathbb{R} \\ \quad t \longmapsto x(t) \end{array} \;\middle|\; \underbrace{x \text{ is } k\text{-times continuously differentiable}}_{\dot{x},\, \ddot{x},\, \dots,\, x^{(k)} \text{ continuous functions}} \right\}$$

$$\dot{x} = x' = \frac{dx}{dt}$$

Ordinary differential equation: $F(t, x, \dot{x}, \dots, x^{(k)}) = 0$

$\boxed{\text{ODE}}$

continuous

Example: $t + x + 2\dot{x} + (\ddot{x})^2 = 0$

(explicit) ODE of order 1: $\dot{x} = w(t, x)$, $w: I \times J \longrightarrow \mathbb{R}$, $I, J \subseteq \mathbb{R}$ intervals

Example: $\dot{x} = x + t$

What about? $\left(\begin{array}{l} \dot{x}_1 = x_2 + t \\ \dot{x}_2 = x_1 + t \end{array} \right) \rightsquigarrow \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = w\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

System of (explicit) ODEs of order 1:

$$\dot{x} = w(t, x) \quad , \quad x(t) \in \mathbb{R}^n \quad , \quad w: I \times U \longrightarrow \mathbb{R}^n$$

open set in $\mathbb{R}^n$

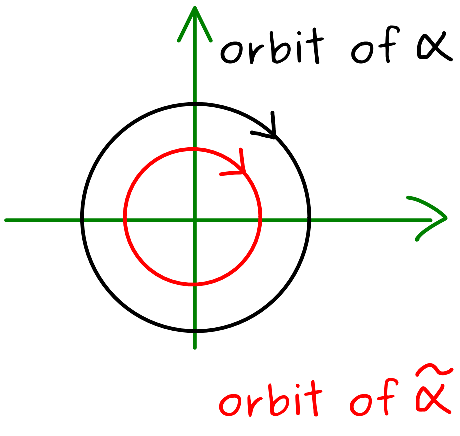
Solution of ODE: $\alpha: (t_0, t_1) \longrightarrow U$ with $(t_0, t_1) \subseteq I$

satisfies $\dot{\alpha}(t) = w(t, \alpha(t))$ for all $t \in (t_0, t_1)$.

Example: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$, $n = 2$, $U = \mathbb{R}^2$, $w(t, x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$

$\rightsquigarrow$ $\alpha(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ is a solution

$\tilde{\alpha}(t) = \frac{1}{2} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ is a solution



Ordinary Differential Equations - Part 3

ODE: $\dot{x} = w(t,x)$ (explicit, of first order)

Example: (a) $\dot{x} = \lambda \cdot x$ ⇝ autonomous

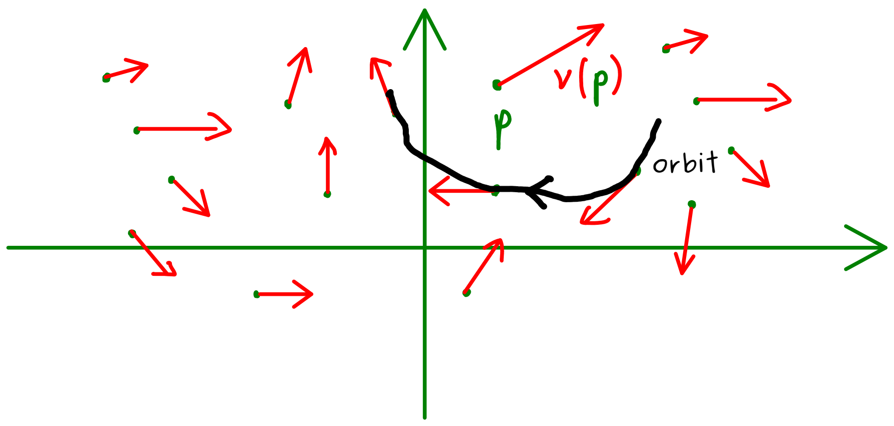
(b) $\dot{x} = t$ ⇝ not autonomous

(c) $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$ ⇝ autonomous

Definition: autonomous system: $\dot{x} = v(x)$ with $v : \underset{U \subseteq \mathbb{R}^n}{U} \rightarrow \mathbb{R}^n$

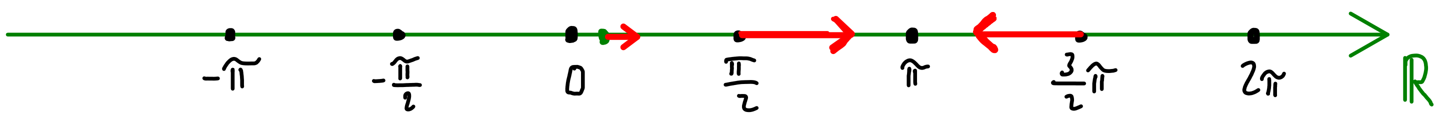
often:
U open
v continuous

Directional field:



$v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Examples: (a) $\dot{x} = \sin(x)$, $v : \mathbb{R} \rightarrow \mathbb{R}$, $v(x) = \sin(x)$



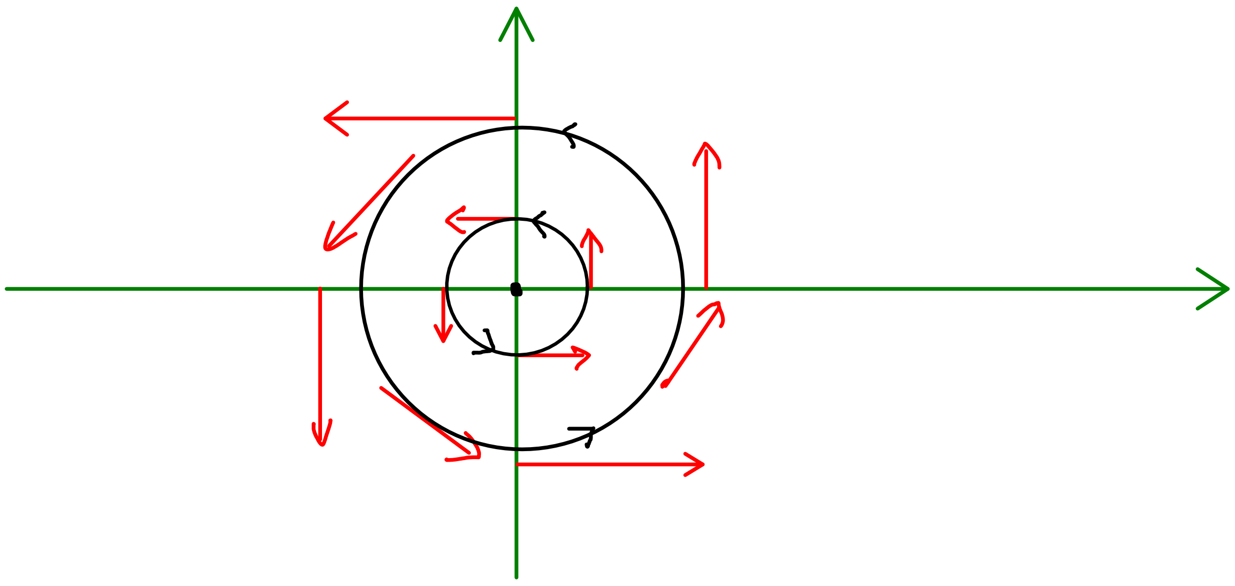
(1) $\alpha(t) = 0$ for all $t \in \mathbb{R}$ is a solution: $\underbrace{\dot{\alpha}(t)}_{=0} = \underbrace{\sin(\alpha(t))}_{=0}$

(2) $\alpha(t) = \pi$ for all $t \in \mathbb{R}$ is a solution.

(3) A solution with $\alpha(0) = \frac{\pi}{2}$ is monotonically increasing

with $\lim_{t \to \infty} \alpha(t) = \pi$.

(b) $\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$, $V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x_1, x_2) \longmapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$



Ordinary Differential Equations – Part 4

Example: $\dddot{x} = \cos(\ddot{x}) + \dot{x}^2 + x$ (autonomous ODE of third order)

define: $y = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \rightsquigarrow$

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= \cos(y_3) + y_2^2 + y_1 \end{aligned}$$

$\Rightarrow \quad \dot{y} = v(y)$ (autonomous system of ODEs of first order)

Example: $\dddot{x} = \cos(\ddot{x}) + \dot{x}^2 + x - t^4$ (non-autonomous ODE of third order)

define: $\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \rightsquigarrow$

$$\begin{aligned} \dot{y}_0 &= 1 \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= \cos(y_3) + y_2^2 + y_1 - y_0^4 \end{aligned}$$

Remember: (explit) autonomous ODE of n th order $\leftrightsquigarrow$ $\underset{\nwarrow \; n \text{ components}}{\dot{y}} = v(y)$

(autonomous system of n ODEs of first order)

(explit) non-autonomous ODE of n th order $\leftrightsquigarrow$ $\underset{\nwarrow \; n+1 \text{ components}}{\dot{y}} = v(y)$

(autonomous system of $n+1$ ODEs of first order)

Ordinary Differential Equations - Part 5

Initial value problem: $\dot{x} = v(x)$ with $v: \mathbb{R} \to \mathbb{R}$ continuous

$x(0) = x_0$

Find all solutions $\alpha : \underset{\ni 0}{(t_0, t_1)} \longrightarrow \mathbb{R}$ $\big(\dot{\alpha}(t) = v(\alpha(t)) \big)$

with $\alpha(0) = x_0$

Solving strategy: Assume $v(x_0) \neq 0$:

ODE: $\dfrac{\dot{x}}{v(x)} = 1$

Therefore: any solution $\alpha : \underset{\ni 0}{(t_0, t_1)} \longrightarrow \mathbb{R}$ with $\alpha(0) = x_0$ satisfies:

$$\frac{\dot{\alpha}(s)}{v(\alpha(s))} = 1 \quad \text{for all } s \in (t_0, t_1)$$

fundamental theorem of calculus

$$\iff \int_0^t \frac{\dot{\alpha}(s)}{v(\alpha(s))} \, ds = t \quad \text{for all } t \in (t_0, t_1)$$

substitution: $x = \alpha(s)$, $dx = \dot{\alpha}(s) \, ds$

$$\iff \int_{x_0}^{\alpha(t)} \frac{1}{v(x)} \, dx = t \quad \text{for all } t \in (t_0, t_1)$$

$$\iff F(\alpha(t)) - F(x_0) = t \quad \text{for all } t \in (t_0, t_1)$$

where F is an antiderivative of $\frac{1}{v}$

$$\iff F(\alpha(t)) = t - c \quad \text{for all } t \in (t_0, t_1)$$

$$\iff \alpha(t) = F^{-1}(t - c) \quad \text{for all } t \in (t_0, t_1)$$

Examples:

(a) $\dot{x} = \lambda \cdot x \quad , \quad x(0) = x_0 \neq 0$

$$\iff \frac{dx}{dt} = \lambda \cdot x \quad \overset{\text{informally}}{\iff} \quad \int \frac{dx}{x} = \int \lambda \, dt$$

$$\iff \log(|x|) = \lambda \cdot t + C \quad , \quad C \in \mathbb{R}$$

natural logarithm

$$\iff |\alpha(t)| = e^{\lambda t} \cdot e^{C}$$

$$\iff \alpha(t) = \begin{cases} -e^{C} e^{\lambda t} \\ e^{C} e^{\lambda t} \end{cases}$$

Solution: $\alpha(t) = x_0 \cdot e^{\lambda t}$

(b) $\dot{x} = x^2 \quad , \quad x(0) = x_0 \neq 0$

$$\iff \frac{dx}{dt} = x^2 \iff \int \frac{dx}{x^2} = \int dt$$

$$\iff -\frac{1}{x} = t + C \quad , \quad C \in \mathbb{R}$$

$$\iff -\frac{1}{\alpha(t)} = t + C \quad , \quad C \in \mathbb{R}$$

$$\iff \alpha(t) = \frac{-1}{t + C} \quad , \quad C \in \mathbb{R}$$

initial value: $\alpha(0) = \frac{-1}{C} \overset{!}{=} x_0 \implies \boxed{C = -\frac{1}{x_0}}$

solution: $\alpha(t) = \dfrac{-1}{t + \left(-\frac{1}{x_0}\right)} = \dfrac{x_0}{1 - x_0 t}$

Ordinary Differential Equations – Part 6

non-autonomous ODE: $\dot{x} = w(t, x)$ can we separate t and x?

example: $\dot{x} = \underbrace{t^3}_{\text{only } t} \cdot \underbrace{x^2}_{\text{only } x}$

Separation of variables: $\dot{x} = g(t) \cdot h(x)$, $x(t_0) = x_0$ (initial value problem)

continuous functions

Assume: $h(x_0) \neq 0$ $\Longrightarrow$ $\dfrac{\dot{x}}{h(x)} = g(t)$

Therefore: any solution $\alpha : \underset{\ni t_0}{(t_1, t_2)} \longrightarrow \mathbb{R}$ with $\alpha(t_0) = x_0$ satisfies:

$$\frac{\dot{\alpha}(s)}{h(\alpha(s))} = g(s) \qquad \text{for all } s \in (t_1, t_2)$$

fundamental theorem of calculus

$$\Longleftrightarrow \int_{t_0}^{t} \frac{\dot{\alpha}(s)}{h(\alpha(s))} \, ds = \int_{t_0}^{t} g(s) \, ds \qquad \text{for all } t \in (t_1, t_2)$$

substitution: $x = \alpha(s)$, $dx = \dot{\alpha}(s) \, ds$

$$\Longleftrightarrow \int_{x_0}^{\alpha(t)} \frac{1}{h(x)} \, dx = \int_{t_0}^{t} g(s) \, ds \qquad \text{for all } t \in (t_1, t_2)$$

$$\Longleftrightarrow F(\alpha(t)) - F(x_0) = G(t) - G(t_0) \qquad \text{for all } t \in (t_1, t_2)$$

where F is an antiderivative of $\frac{1}{h}$

where G is an antiderivative of g

$$\Longleftrightarrow F(\alpha(t)) = G(t) + c$$

for a constant $c \in \mathbb{R}$, for all $t \in (t_1, t_2)$

$$\Longleftrightarrow \alpha(t) = F^{-1}\big(G(t) + c\big)$$

Example: (a) $\dot{x} = \frac{1}{3}t^3 x \ , \ x(0) = x_0 \neq 0$

$$\iff \frac{dx}{dt} = \frac{1}{3}t^3 x \overset{\text{informally}}{\iff} \int \frac{dx}{x} = \int \frac{1}{3}t^3 \, dt$$

$$\iff \log(|x|) = \frac{1}{12}t^4 + c \qquad \text{for a constant } c \in \mathbb{R}$$

natural logarithm

$$\iff |\alpha(t)| = e^{\frac{1}{12}t^4 + c} \overset{\alpha(0) = x_0}{\implies} \alpha(t) = x_0 \cdot e^{\frac{1}{12}t^4}$$

(b) $\dot{x} = \sin(t) \cdot e^{x} \ , \ x(0) = x_0$

$$\iff \frac{dx}{dt} = \sin(t) \cdot e^{x} \overset{\text{informally}}{\iff} \int \frac{dx}{e^{x}} = \int \sin(t) \, dt$$

$$\iff -e^{-x} = -\cos(t) + c \qquad \text{for a constant } c \in \mathbb{R}$$

$$\iff \alpha(t) = -\log(\cos(t) + \tilde{c}) \qquad \text{for a constant } \tilde{c} \in \mathbb{R}$$

$$\overset{\alpha(0) = x_0}{\implies} -\log(\cos(0) + \tilde{c}) = x_0 \implies \tilde{c} = e^{-x_0} - 1$$

Ordinary Differential Equations – Part 7

continuous functions

Linear ODE of first order: $\dot{x} = a(t)\cdot x + b(t)$

Finding solutions: (with an integrating factor)

$$\dot{x} + \tilde{a}(t)\, x = b(t) \qquad \text{with } \tilde{a}(t) := -a(t)$$

$$\overset{\text{multiplying both sides}}{\iff} \quad \dot{x}\, e^{\tilde{A}(t)} + \tilde{a}(t)\, x\, e^{\tilde{A}(t)} = b(t) e^{\tilde{A}(t)}$$

Note: if $\tilde{A}$ is an antiderivative of $\tilde{a}$,

then: $\frac{d}{dt} e^{\tilde{A}(t)} = \underbrace{\tilde{A}'(t)}_{\tilde{a}(t)} \cdot e^{\tilde{A}(t)}$

$$\overset{\text{product rule}}{\iff} \quad \frac{d}{dt}\left(x(t)\, e^{\tilde{A}(t)} \right) = \underbrace{b(t) e^{\tilde{A}(t)}}$$

H is antiderivative of $b(t)e^{\tilde{A}(t)}$

$$\overset{\text{antiderivative}}{\iff} \quad x(t)\, e^{\tilde{A}(t)} = H(t) + c \quad , \quad c \in \mathbb{R}$$

$$\text{Solutions:} \quad \alpha(t) = e^{-\tilde{A}(t)} \left(H(t) + c \right) \quad , \quad c \in \mathbb{R}$$

Example: $\dot{x} = t\, x + e^{\frac{1}{2}t^2} \quad , \quad x(0) = x_0$

$$\iff \quad \dot{x} - t\, x = e^{\frac{1}{2}t^2} \quad \Big| \cdot e^{-\frac{1}{2}t^2}$$

$$\iff \quad \dot{x}\cdot e^{-\frac{1}{2}t^2} - t\, x\, e^{-\frac{1}{2}t^2} = 1$$

$$\iff \quad \frac{d}{dt}\left(x(t)\cdot e^{-\frac{1}{2}t^2} \right) = 1$$

$$\iff \quad x(t)\cdot e^{-\frac{1}{2}t^2} = t + c \quad , \quad c \in \mathbb{R}$$

$$\iff \quad \text{solution:} \quad \alpha(t) = (t + c)\cdot e^{\frac{1}{2}t^2}$$

Initial value condition: $\underset{c}{\underset{\parallel}{\alpha(0)}} = x_0 \quad \rightsquigarrow \quad \alpha(t) = (t + x_0)\cdot e^{\frac{1}{2}t^2}$

Ordinary Differential Equations – Part 8

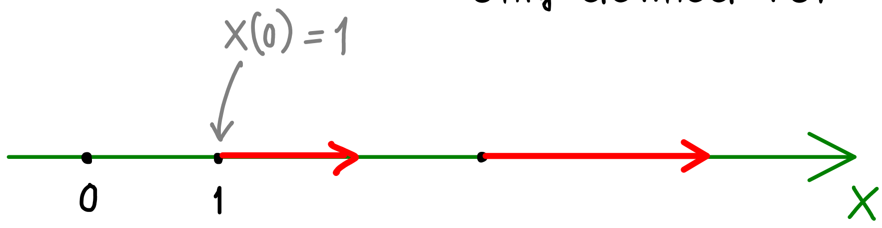
Questions: Initial value problem: $\dot{x} = v(x)$ with $v: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous

$$x(0) = x_0$$

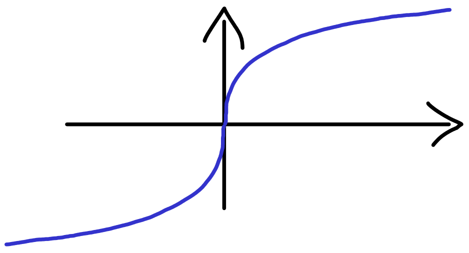
- Does a solution exist?
- What is the domain of definition?
- Uniqueness of solutions?

Examples: (a) $\dot{x} = x^2$, $x(0) = 1$ $\overset{\text{part 5}}{\Longrightarrow}$ solution exists: $\alpha(t) = \dfrac{1}{1-t}$

only defined for $t<1$



(b) $\dot{x} = v(x)$, $x(0) = 0$ with $v(x) = \begin{cases} \sqrt{|x|}, & x \geq 0 \\ -\sqrt{|x|}, & x < 0 \end{cases}$

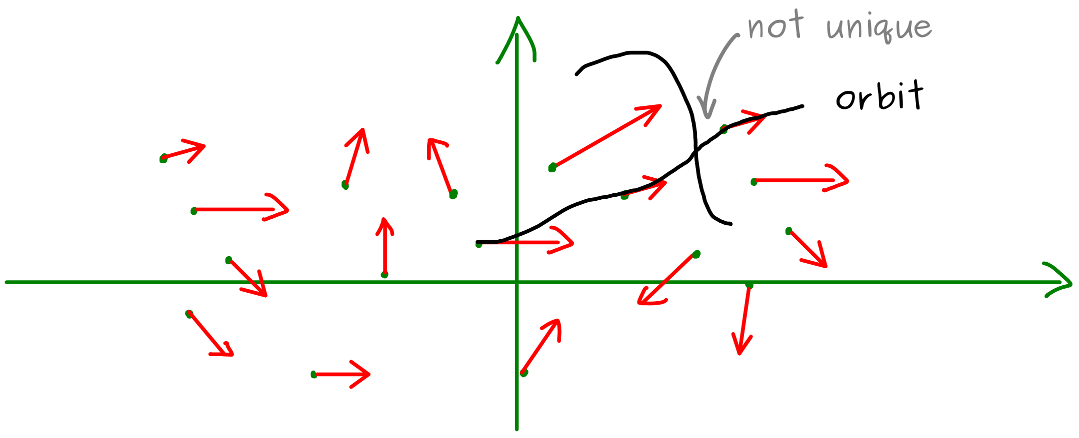


We find at least two solutions: $\alpha(t) = 0$ for all t

$$\tilde{\alpha}(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{4}t^2, & t > 0 \end{cases}$$

In general:

directional field



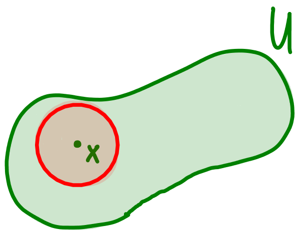
existence: does each point have an orbit?

uniqueness: can two orbits cross?

Ordinary Differential Equations — Part 9

$f: \mathbb{R} \longrightarrow \mathbb{R}$

f continuously differentiable $\Longrightarrow$ f locally Lipschitz continuous $\Longrightarrow$ f continuous



(or open set U)

Definition: $v: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called locally Lipschitz continuous if:

$$\forall_{x \in \mathbb{R}^n} \quad \exists_{\varepsilon > 0} \quad \exists_{L \geq 0} \quad \forall_{y, z \in B_\varepsilon(x)} \; :$$

Lipschitz constant

$$\| v(y) - v(z) \| \leq L \cdot \| y - z \|$$

standard norm of $\mathbb{R}^n$

Remember: (1) v loc. Lipschitz continuous $\Longrightarrow$ v continuous

$$\left(y_n \xrightarrow{n \to \infty} y \;\Rightarrow\; \| v(y_n) - v(y) \| \xrightarrow{n \to \infty} 0 \right)$$

(2) v loc. Lipschitz continuous $\Longrightarrow$ $\dfrac{\| v(y) - v(z) \|}{\| y - z \|} \leq L$

(3) $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuously differentiable. Fix $x \in \mathbb{R}$, $\varepsilon > 0$

$$\frac{| f(y) - f(z) |}{| y - z |} = | f'(\xi) | \qquad \xi \text{ between } y \text{ and } z$$

mean value theorem

$$\leq \sup_{\tilde{\xi} \in B_\varepsilon(x)} | f'(\tilde{\xi}) | =: L \geq 0$$

$\Longrightarrow$ f loc. Lipschitz continuous

Ordinary Differential Equations - Part 10

Initial value problem: $\dot{x} = v(x)$ with $v: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous

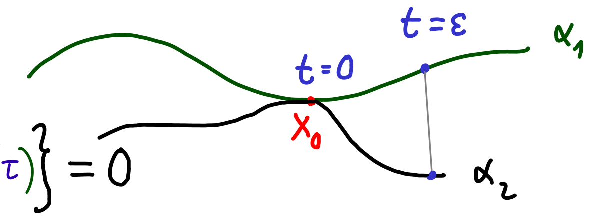
$x(0) = x_0$

Theorem: The initial value problem has at most one solution. (orbits don't cross!)

Proof: Assume α_1, α_2 are two distinct solutions $\big(\alpha_1(0) = \alpha_2(0) = x_0\big)$

with $\alpha_1(\varepsilon) \neq \alpha_2(\varepsilon)$ for $\varepsilon > 0$ and

$\inf\{\tau \in [0,\varepsilon] \mid \alpha_1(\tau) \neq \alpha_2(\tau)\} = 0$



$$\|\beta(t)\| = \|\alpha_1(t) - \alpha_2(t)\|$$

$$= \Big\| \int_0^t \dot{\alpha}_1(\tau)\, d\tau - \int_0^t \dot{\alpha}_2(\tau)\, d\tau \Big\| = \Big\| \int_0^t v(\alpha_1(\tau))\, d\tau - \int_0^t v(\alpha_2(\tau)) d\tau \Big\|$$

$$= \Big\| \int_0^t \big(v(\alpha_1(\tau)) - v(\alpha_2(\tau)) \big) d\tau \Big\| \leq \int_0^t \underbrace{\| v(\alpha_1(\tau)) - v(\alpha_2(\tau)) \|}_{\leq L \cdot \|\alpha_1(\tau) - \alpha_2(\tau)\|} d\tau$$

$$\leq L \cdot \int_0^\varepsilon \|\beta(\tau)\| d\tau \overset{\leq \max \|\beta(\tau)\|}{\leq} \underbrace{L \cdot \varepsilon}_{} \cdot \max_{\tau \in [0,\varepsilon]} \|\beta(\tau)\|$$

choose ε such that $L\varepsilon \leq \frac{1}{2}$

$$\Rightarrow \boxed{\|\beta(t)\| \leq \frac{1}{2} \cdot \max_{\tau \in [0,\varepsilon]} \|\beta(\tau)\| \quad \text{for all } t \in (0,\varepsilon]} \Rightarrow \alpha_1 = \alpha_2 \ \lightning$$

Ordinary Differential Equations – Part 11

$$\boxed{\begin{array}{l}\dot{x} = v(x)\\ x(0) = x_0\end{array}}$$

initial value problem

with $v: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous

integrating

$$\underbrace{\int_0^t \dot{x}(s)\,ds}_{x(t) - x(0)} = \int_0^t v(x(s))\,ds$$

$$\Longrightarrow \quad x(t) = x_0 + \underbrace{\int_0^t v(x(s))\,ds}_{\Phi(x)}$$

function

<u>Definition:</u> $\Phi : \mathcal{F}(\mathbb{R}, \mathbb{R}^n) \longrightarrow \mathcal{F}(\mathbb{R}, \mathbb{R}^n)$

$$f \longmapsto \left(t \mapsto x_0 + \int_0^t v(f(s))\,ds \right)$$

<u>Now we know:</u> $x: \mathbb{R} \longrightarrow \mathbb{R}^n$ is a solution of $\boxed{\begin{array}{l}\dot{x} = v(x)\\ x(0) = x_0\end{array}}$

$$\Longleftrightarrow \quad \Phi(x) = x$$

(fixed point equation)

<u>Banach fixed-point theorem:</u> Let (X, d) be a complete metric space (set with distance function)

and $\Phi : X \longrightarrow X$ be a contraction, which means:

$$\exists\, q \in [0,1) \quad \forall x, \tilde{x} \in X: \quad d(\Phi(x), \Phi(\tilde{x})) \leq \underset{<1}{q} \cdot d(x, \tilde{x}) \quad .$$

Then: Φ has a unique fixed point $x^* \in X$ and

for each $x_0 \in X$ we have: $\Phi^n(x_0) \xrightarrow{n \to \infty} x^*$.

Ordinary Differential Equations – Part 12

initial value problem:

$$\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}} \quad \text{with } v: \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ loc. Lipschitz continuous}$$

$\Longrightarrow$ there is a unique solution!

(Picard-Lindelöf theorem)

Banach fixed-point theorem: Let (X, d) be a complete metric space

and $\Phi : X \longrightarrow X$ be a contraction.

Then: Φ has a unique fixed point $x^* \in X$.

We need:

(1) Complete metric space consisting of functions.

(2) Contraction $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$

Now we know: $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a solution of $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$

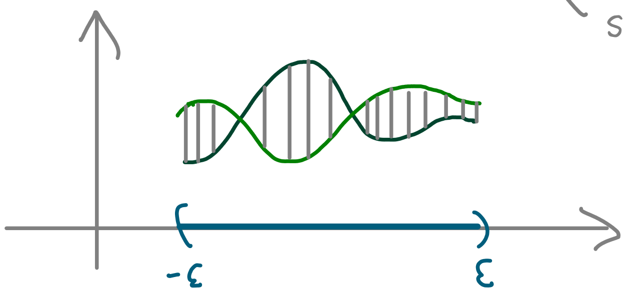
$\Longleftrightarrow \Phi(\alpha) = \alpha$ (fixed point equation)

For (1):

$$X = \left\{ \alpha : (-\varepsilon, \varepsilon) \longrightarrow \tilde{U} \subseteq \mathbb{R}^n \text{ in the domain of } v \text{ with property } (*) \text{ (see below)} \;\middle|\; \alpha \text{ continuous + bounded}, \; \alpha(0) = x_0 \right\}$$

with metric: $$d(\alpha, \beta) := \sup_{t \in (-\varepsilon, \varepsilon)} \left\| \alpha(t) - \beta(t) \right\|_{\mathbb{R}^n}$$

standard norm



Fact: (X, d) is a complete metric space.

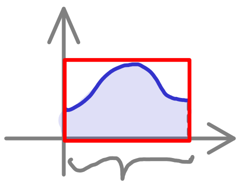
For (2): $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$ gives a map $\Phi : X \longrightarrow X$

$$d(\Phi(\alpha), \Phi(\beta)) = \sup_{t \in (-\varepsilon, \varepsilon)} \left\| \Phi(\alpha)(t) - \Phi(\beta)(t) \right\|_{\mathbb{R}^n}$$

$$= \sup_{t \in (-\varepsilon, \varepsilon)} \left\| \int_0^t \big(v(\alpha(s)) - v(\beta(s)) \big)\, ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\leq \sup_{t \in (-\varepsilon, \varepsilon)} \int_0^t \left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n} ds$$



$$\leq \sup_{t \in (-\varepsilon, \varepsilon)} \underbrace{\text{length}([0,t])}_{|t| \leq \varepsilon} \cdot \underbrace{\sup_{s \in [0,t]} \left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n}}_{\leq \sup_{s \in (-\varepsilon, \varepsilon)} \cdots}$$

$$\leq \varepsilon \cdot \sup_{s \in (-\varepsilon, \varepsilon)} \underbrace{\left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n}}_{\leq L \left\| \alpha(s) - \beta(s) \right\|_{\mathbb{R}^n}}$$

(*) needed

$$\leq \underbrace{\varepsilon \cdot L}_{< 1 \text{ for } \varepsilon \text{ small enough}} \cdot d(\alpha, \beta)$$

contraction

Picard-Lindelöf theorem

$v : U \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous , $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U$

for the initial value problem $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$.

Definition of $\widetilde{\mathcal{U}}$ with property $(*)$

V being locally Lipschitz continuous at x_0 means:

$$\underset{\delta>0}{\exists}\quad \underset{L\geq 0}{\exists}\quad \underset{y,z\in\mathcal{B}_\delta(x)}{\forall} \;:\; \|v(\underset{\substack{\uparrow\\ \alpha(s)}}{y}) - v(\underset{\substack{\uparrow\\ \beta(s)}}{z})\| \leq L\cdot\|y-z\|$$

So we need $\alpha(s), \beta(s) \in \mathcal{B}_\delta(x)$ for all $s\in(-\varepsilon,\varepsilon)$.

Hence: $\widetilde{\mathcal{U}} := \mathcal{B}_\delta(x)$ (not a problem for the solution since we choose ε as small as we want)

Ordinary Differential Equations - Part 13

Picard-Lindelöf theorem

$v: U \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous , $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U$

for the initial value problem $\boxed{\begin{aligned} \dot{x} &= v(x) \\ x(0) &= x_0 \end{aligned}}$.

Picard iteration:

Iteration from the Banach fixed-point theorem $\Phi^n(\tilde{\alpha}) \xrightarrow{n\to\infty} \alpha$

$$\boxed{\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t v(\tilde{\alpha}(s))\, ds}$$

Example: initial value problem: $\begin{aligned} \dot{x} &= x \\ x(0) &= 1 \end{aligned}$

start with $\tilde{\alpha} : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$, $\tilde{\alpha}(t) = 1$

first step: $\Phi(\tilde{\alpha})(t) = 1 + \int_0^t \tilde{\alpha}(s)\, ds = 1 + t$

second step: $\Phi^2(\tilde{\alpha})(t) = 1 + \int_0^t (1+s)\, ds = 1 + t + \frac{1}{2}t^2$

nth step: $\Phi^n(\tilde{\alpha})(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots + \frac{1}{n!}t^n$

$\Big\downarrow n \to \infty$ (pointwise limit) (also uniform limit)

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} = \exp(t)$$

Ordinary Differential Equations – Part 14

initial value problem: $(\mathrm{IVP}^{t_0}_{x_0})$ $\boxed{\begin{array}{c} \dot{x} = v(x) \\ x(t_0) = x_0 \end{array}}$ with $v: \mathcal{D} \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous

$\mathcal{D}$ open in $\mathbb{R}^n$

(Picard-Lindelöf theorem)

$\Longrightarrow$ there is $\varepsilon > 0$ and a unique solution

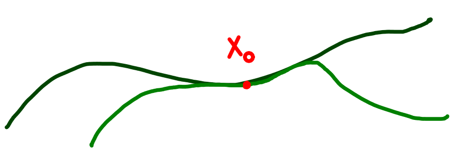
$$\alpha : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{D}$$

Extension of solution: We say a solution $\tilde{\alpha} : I \longrightarrow \mathcal{D}$ extends $\alpha : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{D}$ if $I \supsetneq (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\tilde{\alpha}\big|_{(t_0 - \varepsilon, t_0 + \varepsilon)} = \alpha$.

Maximal solutions: A solution $\alpha : I \longrightarrow \mathcal{D}$ is called maximal if there is no extension.

Proposition: $(\mathrm{IVP}^{t_0}_{x_0})$ for $v: \mathcal{D} \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous has exactly one maximal solution (defined on an open interval).

Proof:



$\alpha_1 : I_1 \longrightarrow \mathcal{D}$

$\alpha_2 : I_2 \longrightarrow \mathcal{D}$

two solutions of $(\mathrm{IVP}^{t_0}_{x_0})$

$\Longrightarrow$ $I := I_1 \cap I_2 = (a, b)$

$\Longrightarrow$ $\alpha_1\big|_I , \alpha_2\big|_I$ two solutions of $(\mathrm{IVP}^{t_0}_{x_0})$

There is $\varepsilon > 0$ such that $\alpha_1\big|_{(t_0 - \varepsilon, t_0 + \varepsilon)} = \alpha_2\big|_{(t_0 - \varepsilon, t_0 + \varepsilon)}$

$$\left\{ J \text{ open interval} \;\middle|\; I \subseteq J \subseteq (t_0 - \varepsilon, t_0 + \varepsilon) \text{ with } \alpha_1\big|_J = \alpha_2\big|_J \right\} = \mathcal{M}$$

$(t_-, t_+) := \bigcup_{J \in \mathcal{M}} J$ gives maximal open interval

Show: $t_+ = b$ **Assume:** $t_+ \neq b$

Then:

$$\begin{array}{rcccl} & \alpha_1(t) & = & \alpha_2(t) & \text{for all } t \in (t_-, t_+) \\ & \big\downarrow & {\scriptstyle t \to t_+} & \big\downarrow & \\ \tilde{x}_0 = & \alpha_1(t_+) & = & \alpha_2(t_+) & \text{because of continuity on } I \end{array}$$

Look at $(IVP^{t_+}_{\tilde{x}_0})$: uniqueness of solution implies:

$$\alpha_1(t) = \alpha_2(t) \text{ for } t \in (t_+ - \tilde{\varepsilon}, t_+ + \tilde{\varepsilon})$$

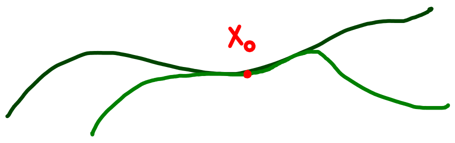
$$\Rightarrow (t_-, t_+ + \tilde{\varepsilon}) \in \mathcal{M} \ ↯$$

Conclusion: $(t_-, t_+) = I$ and

$$\alpha_1|_I = \alpha_2|_I \Rightarrow \alpha : I_1 \cup I_2 \longrightarrow D$$

Define: $\left\{ I \text{ open interval} \;\middle|\; \text{there is a solution } \alpha : I \longrightarrow D \text{ for } (IVP^{t_0}_{x_0}) \right\} = \mathcal{S}$

$\bigcup_{I \in \mathcal{S}} I$ open interval for maximal solution ☐



cannot happen!

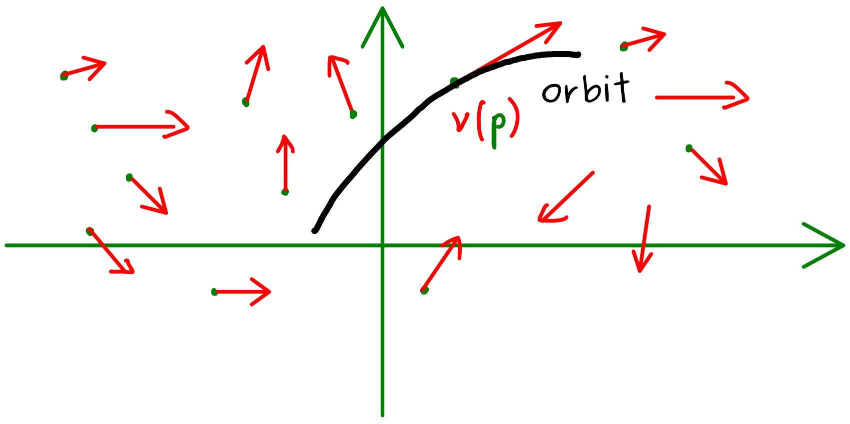
Definition: If the maximal solution is defined on $I = \mathbb{R}$, then it's called a global solution.

Ordinary Differential Equations - Part 15

$\dot{x} = v(x)$ vector field

$v: \mathcal{D} \longrightarrow \mathbb{R}^n$

open in $\mathbb{R}^n$



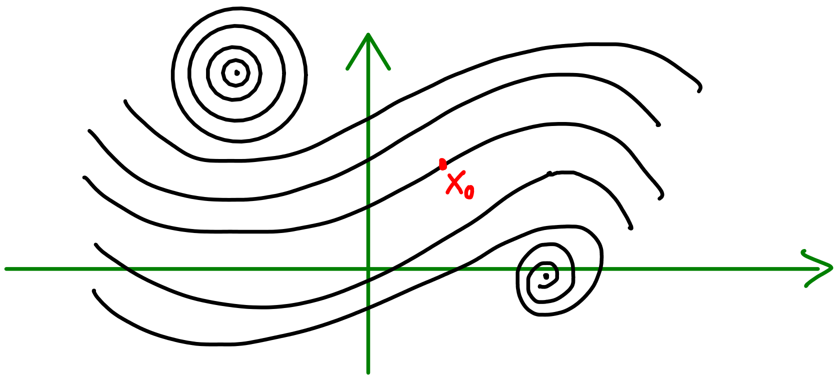
For $v: \mathcal{D} \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous :

$(\mathrm{IVP}^{t_0}_{x_0})$ $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(t_0) = x_0 \end{array}}$ has a unique maximal solution $\alpha: I \longrightarrow \mathcal{D}$

$$\Updownarrow \quad \beta(\tilde{t}) := \alpha(\tilde{t} + t_0)$$

$\beta: \tilde{I} \longrightarrow \mathcal{D}$ is a maximal solution $(\mathrm{IVP}^{0}_{x_0})$

Phase portrait:



orbit at x_0

$$\left\{ \alpha(t) \,\middle|\, t \in I \text{ where } \alpha: I \longrightarrow \mathcal{D} \text{ is the max. solution of } (\mathrm{IVP}^{0}_{x_0}) \right\}$$

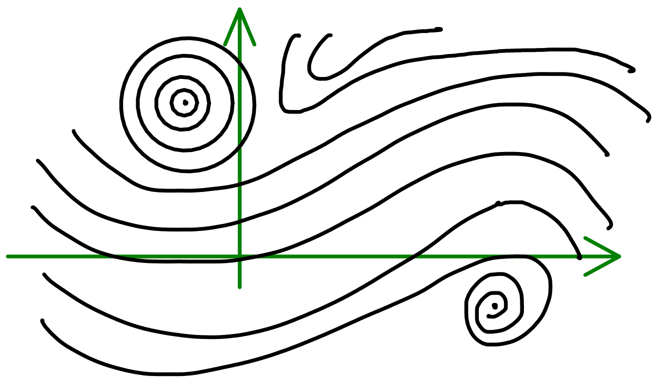
Proposition: For $v: \mathcal{D} \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous, the phase portrait satisfies:

(a) For all $x \in \mathcal{D}$ there is an orbit $\mathcal{O} \ni x$.

(b) Two orbits $\mathcal{O}_1, \mathcal{O}_2$ satisfy: $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset \Rightarrow \mathcal{O}_1 = \mathcal{O}_2$

Ordinary Differential Equations – Part 16

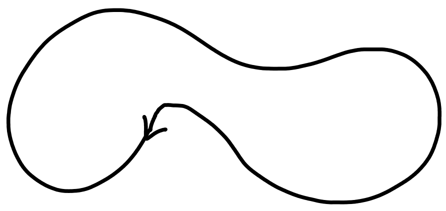
$\dot{x} = v(x)$, $v: \underset{\text{open in } \mathbb{R}^n}{D} \longrightarrow \mathbb{R}^n$



Definition: A global solution $\alpha : \mathbb{R} \longrightarrow D$ of $\dot{x} = v(x)$ is called:

- fixed point if $\alpha(t) = \alpha(0)$ for all $t \in \mathbb{R}$.
- periodic if there is a $T > 0$ with $\alpha(t+T) = \alpha(t)$ for all $t \in \mathbb{R}$.

T: a period



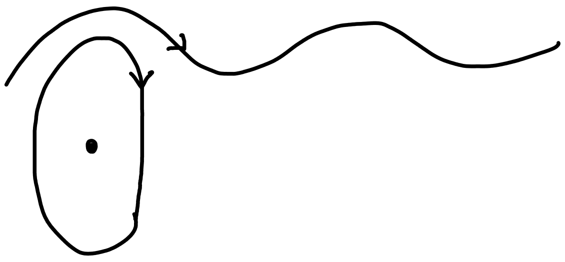
Proposition: For $v: D \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous,

there are three options for the maximal solution α of $\boxed{\begin{matrix}\dot{x} = v(x) \\ x(0) = x_0\end{matrix}}$:

(a) α is injective

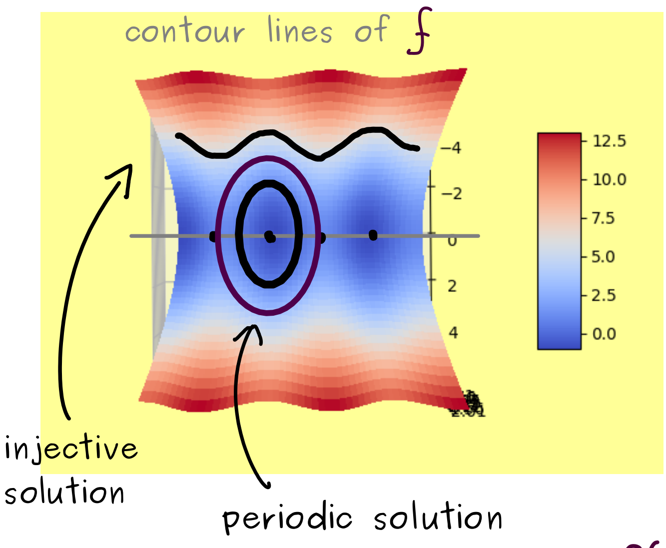
(b) α is fixed point

(c) α is periodic



Example: $\ddot{x} = -\sin(x) \rightsquigarrow \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ -\sin(x_1) \end{pmatrix} = v(x_1, x_2)$

Do we have $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $f(\alpha(t)) =$ constant for all t ?



Note: $f(\alpha(t)) = \text{constant}$ for all t

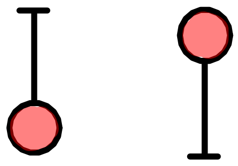
$$\Leftrightarrow \quad \frac{d}{dt} f(\alpha(t)) = 0 \quad \text{for all } t$$

$$\overset{\text{chain rule}}{\Leftrightarrow} \quad \left\langle \operatorname{grad} f(\alpha(t)) \, , \, \underbrace{\dot{\alpha}(t)}_{V(\alpha(t))} \right\rangle = 0 \quad \text{for all } t$$

$f(x_1, x_2) = \frac{1}{2} x_2^2 - \cos(x_1)$ satisfies $\left\langle \operatorname{grad} f(x_1,x_2), V(x_1,x_2) \right\rangle = 0$.

Fixed point: $\operatorname{grad} f(x_1, x_2) = \begin{pmatrix} \sin(x_1) \\ x_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x_2 = 0 \, , \; x_1 = k \cdot \pi$

$k \in \mathbb{Z}$



(pendulum does not move)

Ordinary Differential Equations - Part 17

In part 12: Picard-Lindelöf theorem for initial value problem

$$\boxed{\begin{aligned} \dot{x} &= v(x) \\ x(0) &= x_0 \end{aligned}}$$

v: locally Lipschitz continuous

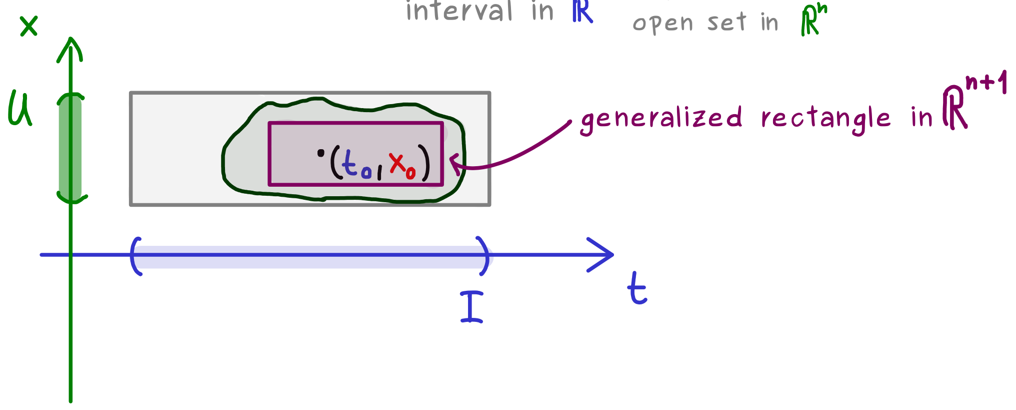
$\Longrightarrow$ there is a unique solution

Now:

$$\boxed{\begin{aligned} \dot{x} &= w(t,x) \\ x(t_0) &= x_0 \end{aligned}}$$

initial value problem

continuous function $w : \underset{\text{interval in } \mathbb{R}}{I} \times \underset{\text{open set in } \mathbb{R}^n}{U} \longrightarrow \mathbb{R}^n$

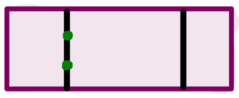


Picard-Lindelöf theorem (for non-autonomous systems)

Assume $w : \underset{\text{interval in } \mathbb{R}}{I} \times \underset{\text{open set in } \mathbb{R}^n}{U} \longrightarrow \mathbb{R}^n$ continuous! satisfies: $\forall K \subseteq I \times U$ compact $\exists L_K > 0 \ \forall (t,x),(t,y) \in K$:

$$\| w(t,x) - w(t,y) \| \leq L_K \cdot \| x - y \|$$

standard norm in $\mathbb{R}^n$



Then: For $x_0 \in U$, there is $\varepsilon > 0$ and a unique solution $\alpha : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow U$

for the initial value problem
$$\boxed{\begin{aligned} \dot{x} &= w(t,x) \\ x(t_0) &= x_0 \end{aligned}} \ .$$

Proof: Same as in part 12 with $\Phi(\alpha)(t) = x_0 + \int_{t_0}^{t} w(s, \alpha(s))\, ds$

and Banach fixed-point theorem.

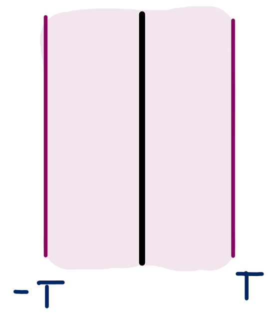
Picard-Lindelöf theorem (special version)

Assume $w : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and satisfies: for each $T > 0$:

$$\exists L_T > 0 \;\; \forall t \in [-T,T] \;\; \forall x,y \in \mathbb{R}^n : \; \left\| w(t,x) - w(t,y) \right\| \leq L_T \cdot \|x - y\|$$

Then there is a unique global solution $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$

for the initial value problem $\boxed{\begin{aligned} \dot{x} &= w(t,x) \\ x(t_0) &= x_0 \end{aligned}}$.



Proof: Set $t_0 = 0$. Complete metric space $X = C([-T,T], \mathbb{R}^n)$

with metric $d(\alpha, \beta) := \sup_{t \in [-T,T]} e^{-2L_T|t|} \cdot \| \alpha(t) - \beta(t) \|_{\mathbb{R}^n}$

standard norm

$$\Phi(\alpha)(t) = x_0 + \int_0^t w(s, \alpha(s))\, ds$$

$$d(\Phi(\alpha), \Phi(\beta)) = \sup_{t \in [-T,T]} e^{-2L_T|t|} \left\| \Phi(\alpha)(t) - \Phi(\beta)(t) \right\|_{\mathbb{R}^n}$$

$$= \sup_{t \in [-T,T]} e^{-2L_T|t|} \left\| \int_0^t \Big(w(s, \alpha(s)) - w(s, \beta(s)) \Big)\, ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\leq \sup_{t \in [-T,T]} e^{-2L_T|t|} \left| \int_0^t \underbrace{\left\| w(s, \alpha(s)) - w(s, \beta(s)) \right\|_{\mathbb{R}^n}}_{\leq L_T \cdot \| \alpha(s) - \beta(s) \|_{\mathbb{R}^n}} ds \right|$$

$$\leq \sup_{t\in[-T,T]} e^{-2L_T|t|} \left| \int_0^t L_T \cdot e^{2L_T|s|} \underbrace{e^{-2L_T|s|} \|\alpha(s) - \beta(s)\|_{\mathbb{R}^n}}_{\leq d(\alpha,\beta)} ds \right|$$

$$\leq \sup_{t\in[-T,T]} e^{-2L_T|t|} L_T \cdot d(\alpha,\beta) \underbrace{\left| \int_0^t e^{2L_T|s|} ds \right|}_{\frac{1}{2L_T}\left(e^{2L_T|t|} - 1\right)}$$

$$\leq \frac{1}{2} d(\alpha,\beta) \underbrace{\sup_{t\in[-T,T]} \left(1 - e^{-2L_T|t|}\right)}_{\leq 1}$$

$$\leq \frac{1}{2} d(\alpha,\beta)$$

$\Longrightarrow$ $\Phi : X \rightarrow X$ is a contraction $\overset{\text{Banach fixed-point theorem:}}{\Longrightarrow}$ unique solution $\alpha : [-T,T] \rightarrow \mathbb{R}^n$

for all $T > 0$

$\Longrightarrow$ global solution $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n$ $\square$

Ordinary Differential Equations - Part 18

Definition: A system of ODEs $\dot{x} = w(t,x)$ is called a system of linear differential equations if

$$\underset{\text{continuous}}{w} : \underset{\text{or interval}}{\mathbb{R}} \times \underset{\text{or open set}}{\mathbb{R}^n} \longrightarrow \mathbb{R}^n , \quad (t,x) \longmapsto A(t)\,x + b(t)$$

with $A : t \mapsto A(t) \in \mathbb{R}^{n \times n}$ continuous

$b : t \mapsto b(t) \in \mathbb{R}^n$ continuous

Note:
- If $b(t) = 0$ for all t, then the system is called homogeneous.
- If $A(t) = A$, $b(t) = b$ for all t, then the system is called autonomous.

Lipschitz condition?

$$\|w(t,x) - w(t,y)\| = \|A(t)x + b(t) - (A(t)y + b(t))\|$$

$$= \|A(t)(x-y)\| \leq \underset{\text{matrix norm/ operator norm}}{\|A(t)\|} \cdot \|x-y\|$$

$[-T,T] \ni t \mapsto \|A(t)\|$ continuous

$$\leq L_T \cdot \|x-y\|$$

Picard-Lindelöf theorem
(special version)

$\Longrightarrow$ unique global solution $\alpha : \underset{\text{or interval}}{\mathbb{R}} \longrightarrow \mathbb{R}^n$ for initial value problem $\boxed{\begin{array}{l} \dot{x} = w(t,x) \\ x(t_0) = x_0 \end{array}}$.

Example:

$$\overset{\bullet}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 2t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix}$$

$$\rightsquigarrow \overset{\bullet}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} - \begin{pmatrix} 1 \cdot x_1 \\ 2t \cdot x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix} \begin{matrix} \leftarrow e^{-t} \\ \leftarrow e^{-t^2} \end{matrix}$$

$$\rightsquigarrow \begin{cases} \dot{x}_1 e^{-t} - x_1 e^{-t} = 0 \\ \dot{x}_2 e^{-t^2} - x_2 \cdot 2t\, e^{-t^2} = 1 \end{cases} \rightsquigarrow \begin{cases} \frac{d}{dt}\left(x_1 e^{-t}\right) = 0 \\ \frac{d}{dt}\left(x_2 e^{-t^2}\right) = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 e^{-t} = C_1 \\ x_2 e^{-t^2} = t + C_2 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} C_1 e^{t} \\ (t + C_2) e^{t^2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & 0 \\ 0 & e^{t^2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t e^{t^2} \end{pmatrix}$$

Ordinary Differential Equations - Part 19

System of linear differential equations: (of first order)

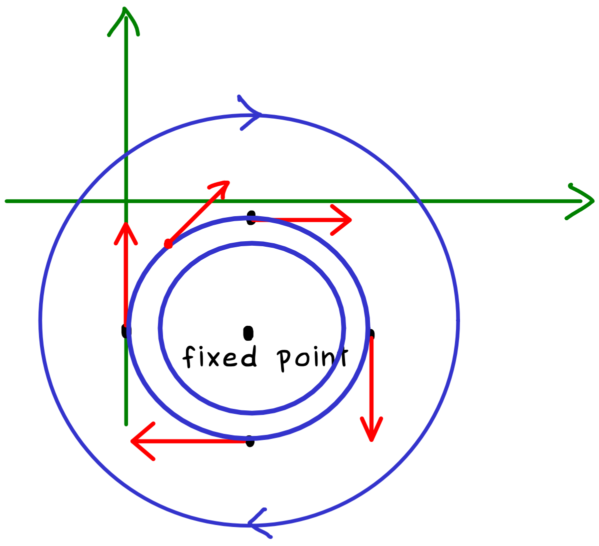
$$\dot{x} = A(t)\,x + b(t) \quad \text{with} \quad \begin{array}{l} I \ni t \longmapsto A(t) \in \mathbb{R}^{n\times n} \\ \qquad\quad \text{continuous} \\ I \ni t \longmapsto b(t) \in \mathbb{R}^{n} \end{array}$$

interval in $\mathbb{R}$

- solutions are global $\alpha : I \to \mathbb{R}^n$
- autonomous systems: $A(t) = A$, $b(t) = b$

example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$v(x) = Ax + b$$



- corresponding homogeneous system

$$\dot{x} = A(t)\,x$$

Fact: If $\alpha : I \to \mathbb{R}^n$, $\beta : I \to \mathbb{R}^n$ are two solutions of $\dot{x} = A(t)\,x$,

$$(\alpha + \beta)^{\cdot}(t) = \dot{\alpha}(t) + \dot{\beta}(t) = A(t)\alpha(t) + A(t)\beta(t)$$

$$= A(t)\big(\alpha(t) + \beta(t)\big)$$

$$(\lambda \cdot \alpha)^{\cdot}(t) = A(t)\big(\lambda \cdot \alpha(t)\big)$$

$\rightsquigarrow$ $\boxed{\text{linear combinations of solutions are solutions}}$

Proposition: The solution set of the corresponding homogeneous system

$$S_0 := \left\{ \alpha : I \longrightarrow \mathbb{R}^n \ \text{continuously differentiable} \ \middle| \ \dot{\alpha}(t) = A(t)\,\alpha(t) \ \underset{t\in I}{\text{for all}} \right\}$$

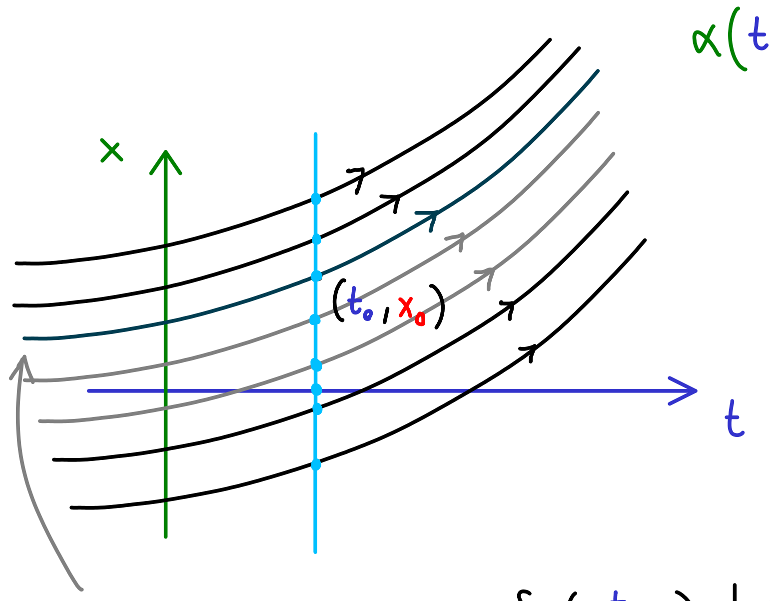
forms an n-dimensional $\mathbb{R}$-vector space.

Proof: S_0 is a subspace in the $\mathbb{R}$-vector space $C^1(I, \mathbb{R}^n)$.

What about the dimension of S_0 ?

$$(\text{IVP}^{t_0}_{x_0}) \quad \boxed{\begin{array}{l} \dot{x} = A(t)\,x \\ x(t_0) = x_0 \end{array}} \quad \xrightarrow[\text{(special version)}]{\text{Picard-Lindelöf theorem}} \quad \text{unique solution } \alpha : I \longrightarrow \mathbb{R}^n, \quad \alpha(t_0) = x_0$$

extended phase portrait:



extended orbit at (t_0, x_0) : $\left\{ \begin{pmatrix} t \\ \alpha(t) \end{pmatrix} \middle| \ \begin{array}{l} t\in I \text{ where } \alpha \text{ is} \\ \text{unique solution of } (\text{IVP}^{t_0}_{x_0}) \end{array} \right\}$

define a map:
$$\ell : S_0 \longrightarrow \mathbb{R}^n$$
$$\alpha \longmapsto \alpha(t_0)$$
linear map!

↳ surjective (every $(\text{IVP}^{t_0}_{x_0})$ has a solution)

↳ injective $\left(\ell(\alpha) = \ell(\beta) \Longrightarrow \alpha(t_0) = \beta(t_0) \overset{\text{uniqueness}}{\Longrightarrow} \alpha = \beta \text{ on } I \right)$

$$\Longrightarrow \ \ell : S_0 \longrightarrow \mathbb{R}^n \ \text{isomorphism}$$

$$\Longrightarrow \ \dim(S_0) = \dim(\mathbb{R}^n) = n$$

□

Ordinary Differential Equations - Part 20

System of linear differential equations (homogeneous + autonomous)

$$(\mathrm{IVP}^{0}_{x_0}) \quad \boxed{\begin{array}{l} \dot{x} = A x \\ x(0) = x_0 \end{array}} \quad , \qquad \begin{array}{l} A \in \mathbb{R}^{n\times n} \\ x_0 \in \mathbb{R}^n \end{array}$$

↪ $\underbrace{\text{Picard iteration (see part 13)}}$

start with a guess $\tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$\boxed{\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t A\,\tilde{\alpha}(s)\,ds} \rightsquigarrow \Phi^n(\tilde{\alpha}) \xrightarrow{n\to\infty} \alpha$$

↑ solution of $(\mathrm{IVP}^{0}_{x_0})$

Picard iteration: guess: $\tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}^n$, $\tilde{\alpha}(t) = x_0$

1st step: $\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t A x_0 \, ds = (\mathbb{1} + tA)\, x_0$

2nd step: $\Phi^2(\tilde{\alpha})(t) = x_0 + \int_0^t A\big((\mathbb{1} + tA)\,x_0\big)\,ds$

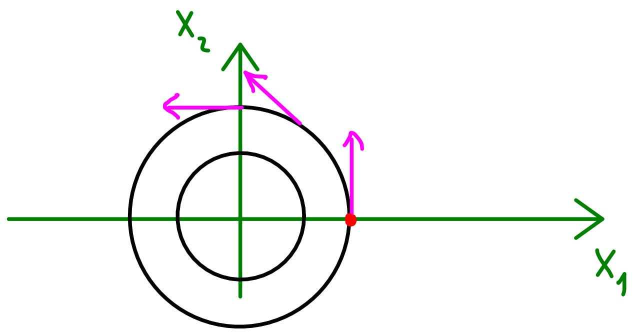
$$= x_0 + tAx_0 + \tfrac{1}{2}t^2A^2x_0 = \left(\mathbb{1} + tA + \tfrac{1}{2}t^2A^2\right)x_0$$

nth step: $\Phi^n(\tilde{\alpha})(t) = \left(\mathbb{1} + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{6}t^3A^3 + \cdots + \tfrac{1}{n!}t^nA^n\right)x_0$

$\xrightarrow{n\to\infty}$ solution of $(\mathrm{IVP}^{0}_{x_0})$ $\quad \alpha(t) = \underbrace{\sum_{k=0}^{\infty} \frac{(t\cdot A)^k}{k!}}_{=:\, \exp(t\cdot A) \,=\, e^{tA}} x_0$

matrix exponential

Example: $$\overset{\bullet}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\exp(t \cdot A) = \Big(\mathbb{1} + t \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\underset{\shortparallel}{A}} + \frac{1}{2} t^2 \underset{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{A^2}} + \frac{1}{6} t^3 \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{\underset{\shortparallel}{A^3}} + \frac{1}{4!} t^4 \underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}}{\underset{\shortparallel}{A^4}} + \cdots \Big)$$

$$= \begin{pmatrix} \underbrace{1 - \frac{1}{2} t^2 + \frac{1}{4!} t^4 \pm \cdots}_{= \cos(t)} & -\sin(t) \\ \underbrace{0 + t - \frac{1}{6} t^3 + \frac{1}{5!} t^5 \pm \cdots}_{= \sin(t)} & \cos(t) \end{pmatrix}$$

solution of $(\mathrm{IVP}^{0}_{x_0})$ with $x_0 = \begin{pmatrix} c \\ 0 \end{pmatrix}$:

$$\alpha(t) = \exp(t \cdot A)\, x_0 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = c \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$\Longrightarrow$ all orbits are circles!

Ordinary Differential Equations - Part 21

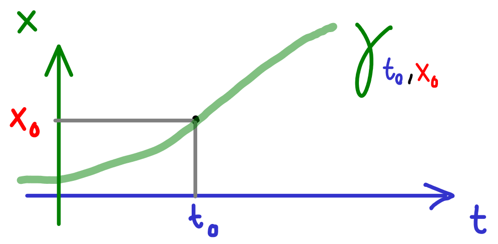
System of linear differential equations: $\dot{x} = A(t)\,x + b(t) \qquad (*)$

with $I \ni t \longmapsto A(t) \in \mathbb{R}^{n\times n}$

interval in $\mathbb{R}$ $\quad I \ni t \xmapsto{\text{continuous}} b(t) \in \mathbb{R}^n$

We already know:

- the homogeneous part of $(*)$ $\left(\dot{x} = A(t)\,x\right)$ has an n-dimensional solution space $\mathcal{S}_0$
- the initial value problem $(\mathrm{IVP}^{t_0}_{x_0})$ $\boxed{\begin{array}{l}\dot{x} = A(t)\,x + b(t)\\ x(t_0) = x_0\end{array}}$ has a global solution $\gamma_{t_0,x_0} : I \longrightarrow \mathbb{R}^n$



Solution set: $\mathcal{S} := \left\{ \beta : I \to \mathbb{R}^n \text{ continuously differentiable} \;\middle|\; \beta \text{ solution of } (*) \right\}$

$$\mathcal{S}_0 + \gamma_{t_0,x_0} := \left\{ \alpha + \gamma_{t_0,x_0} \;\middle|\; \alpha \in \mathcal{S}_0 \right\} \qquad \text{(affine subspace)}$$

Show $\mathcal{S} = \mathcal{S}_0 + \gamma_{t_0,x_0}$ **:** $(\supseteq)$ Take $\alpha \in \mathcal{S}_0$: $A(t)\left(\alpha(t) + \gamma_{t_0,x_0}(t)\right) + b(t)$

$$= \underbrace{A(t)\,\alpha(t)} + \underbrace{A(t)\,\gamma_{t_0,x_0}(t) + b(t)}$$

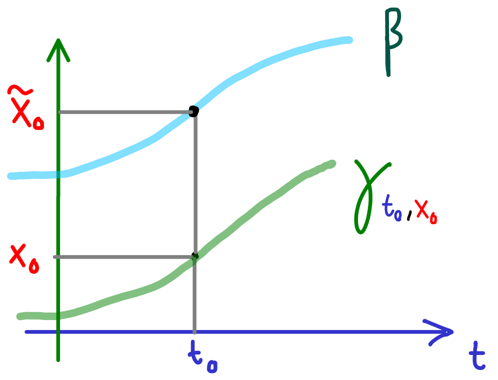
$$= \dot{\alpha}(t) + \dot{\gamma}_{t_0,x_0}(t)$$

$$= \left(\alpha + \gamma_{t_0,x_0}\right)^{\bullet}(t)$$

$$\Longrightarrow \alpha + \gamma_{t_0,x_0} \in \mathcal{S}$$

$(\subseteq)$ Take $\beta \in \mathcal{S}$ and set $\tilde{x}_0 := \beta(t_0)$

$\Rightarrow$ β is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$



Choose $\alpha \in \mathcal{S}_0$ as the solution of the initial value problem $\boxed{\begin{array}{l} \dot{x} = A(t)\,x \\ x(t_0) = \tilde{x}_0 - x_0 \end{array}}$

Then: $\alpha + \gamma_{t_0,x_0} \in \mathcal{S}$ with $(\alpha + \gamma_{t_0,x_0})(t_0) = \alpha(t_0) + \gamma_{t_0,x_0}(t_0)$

$$= \tilde{x}_0 - x_0 + x_0 = \tilde{x}_0$$

$\Rightarrow$ $\alpha + \gamma_{t_0,x_0}$ is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$

$\overset{\text{uniqueness}}{\Rightarrow}$ $\beta = \alpha + \gamma_{t_0,x_0}$ ☐

Result: The solution set of $\dot{x} = A(t)\,x + b(t)$ is given by

$$\mathcal{S} = \mathcal{S}_0 + \gamma$$

where $\mathcal{S}_0$ is the solution space of the homogeneous part $\dot{x} = A(t)\,x$ and γ is a particular solution of $\dot{x} = A(t)\,x + b(t)$.

($\mathcal{S}$ is an n-dimensional affine subspace)

Ordinary Differential Equations - Part 22

$$A \in \mathbb{R}^{n\times n} \implies e^{tA} \in \mathbb{R}^{n\times n}$$ columns span solution space of $\dot{x} = Ax$

Remember: $\boxed{\begin{array}{l} \dot{x} = Ax \\ x(0) = x_0 \end{array}}$ has unique solution: $t \mapsto e^{tA} x_0$

Definition:

$$e^{tA} := \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k}$$ exists for every $t \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$

- each component is a function $t \in [a,b] \longrightarrow \mathbb{R}$
- we have uniform convergence

Properties: (a) derivative of the matrix exponential:

$$\frac{d}{dt} e^{tA} := \lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} \frac{(t+h)^k}{k!} A^k - \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \left((t+h)^k - t^k \right) \right)$$

$$\overset{\text{uniform convergence}}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!} \underbrace{\lim_{h\to 0} \frac{(t+h)^k - t^k}{h}}_{= \begin{cases} k\cdot t^{k-1} & , k \geq 1 \\ 0 & , k = 0 \end{cases}}$$

$$= \sum_{k=1}^{\infty} \frac{A^k}{k!} \cdot k \cdot t^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1} \cdot A = e^{tA} A = A e^{tA}$$

(b) exponentiation identity: $e^{A+B} = e^{A} e^{B}$ for matrices with $AB = BA$

(c) inverse:

$$\left.\begin{array}{l} e^{A} e^{-A} = e^{A-A} = e^{0} = \mathbb{1} \\ e^{-A} e^{A} = e^{A-A} = e^{0} = \mathbb{1} \end{array}\right\} \left(e^{A}\right)^{-1} = e^{-A}$$

Ordinary Differential Equations - Part 23

Example: System of linear differential equations (homogeneous + autonomous)

$$\begin{aligned}\dot{x}_1 &= -x_1 + 3x_2 \\ \dot{x}_2 &= x_1 + x_2\end{aligned} \quad \Longleftrightarrow \quad \dot{x} = \underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}}_{A} x$$

General solution: $e^{tA} = \sum_{k=0}^{\infty} \frac{(t \cdot A)^k}{k!}$

(columns span solution space)

Remark: If $\mathcal{B} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, then $\mathcal{B}^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}$ and $e^{\mathcal{B}} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$

If A is diagonalizable, then $A = X \mathcal{D} X^{-1}$, $A^2 = X \mathcal{D} \underbrace{X^{-1} X}_{\mathbb{1}} \mathcal{D} X^{-1} = X \mathcal{D}^2 X^{-1}$

$$A^k = X \mathcal{D}^k X^{-1}$$

$$\Longrightarrow \quad e^{tA} = X \cdot e^{t\mathcal{D}} X^{-1}$$

Back to the example: $A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$

eigenvalues: $0 = \det(A - \lambda \cdot \mathbb{1}) = \det \begin{pmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 - 4$

$$\Longrightarrow \quad \lambda_1 = -2 \, , \quad \lambda_2 = 2$$

eigenvectors: $\operatorname{Ker}(A - \lambda_1 \cdot \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\operatorname{Ker}(A - \lambda_2 \cdot \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

form invertible matrix: $X = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow X^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$

diagonalization: $A = X D X^{-1}$ with $D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

matrix exponential:

$$e^{tA} = X \cdot e^{tD} X^{-1} = X \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{pmatrix} X^{-1}$$

$$= \frac{1}{4} \begin{pmatrix} 3e^{-2t} + e^{2t} & -3e^{-2t} + 3e^{2t} \\ -e^{-2t} + e^{2t} & e^{-2t} + 3e^{2t} \end{pmatrix}$$

solution of initial value problem:

$$\boxed{\begin{array}{l} \dot{x} = A x \\ x(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \end{array}} \rightsquigarrow x(t) = e^{tA} \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-2t} + 3e^{2t} \\ e^{-2t} + 3e^{2t} \end{pmatrix}$$

Ordinary Differential Equations - Part 24

System of linear ODEs (homogeneous + autonomous)

$$\dot{x} = \overset{(n \times n)\text{- matrix}}{A} x \quad \rightsquigarrow \quad \text{solutions} \quad t \longmapsto \underbrace{e^{tA}}_{} \cdot x_0 \quad \text{for} \quad x_0 \in \mathbb{R}^n$$

↳ easy to calculate if A has n different eigenvalues:

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

Linear ODE of order n (homogeneous + autonomous)

$$x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

Transform into system of first order:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n-1)} \end{pmatrix} \quad \text{and} \quad \dot{y}_n = -a_{n-1} y_n - \cdots - a_1 y_2 - a_0 y_1$$

$$\Longrightarrow \quad \dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ \dot{y}_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}}_{A \quad (n \times n)\text{- matrix}} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

General solution: $e^{tA} \longleftrightarrow$ eigenvalues of A

Characteristic polynomial:

$$\det(A - \lambda \cdot \mathbb{1}) = (-1)^n \underbrace{\left(\lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \cdot \lambda^1 + a_0 \right)}_{}$$

is called the characteristic polynomial of the ODE

$$\boxed{x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0}$$

Rule of thumb: Use approach $x(t) = e^{\lambda t}$ ↗

Result: If the characteristic polynomial has n different zeros in the real numbers

$$\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \text{ , then:}$$

$$e^{\lambda_1 t} \cdot \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{n-1} \end{pmatrix}, \; e^{\lambda_2 t} \cdot \begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \\ \vdots \\ \lambda_2^{n-1} \end{pmatrix}, \dots, \; e^{\lambda_n t} \cdot \begin{pmatrix} 1 \\ \lambda_n \\ \lambda_n^2 \\ \vdots \\ \lambda_n^{n-1} \end{pmatrix}$$

span solution space of

$$\dot{y} = Ay$$

and $t \mapsto e^{\lambda_1 t}$, $t \mapsto e^{\lambda_2 t}$, …, $t \mapsto e^{\lambda_n t}$ span solution space of

$$\boxed{x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \dots + a_1 \dot{x} + a_0 x = 0}$$