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ON STEADY

The Bright Side of Mathematics

Ordinary Differential Equations – Part 12

initial value problem: $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$ with $v: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lipschitz continuous $\Longrightarrow$ there is a unique solution!

(Picard-Lindelöf theorem)

Banach fixed-point theorem: Let (X, d) be a complete metric space and $\Phi : X \to X$ be a contraction.

Then: Φ has a unique fixed point $x^* \in X$.

We need: (1) Complete metric space consisting of functions.

(2) Contraction $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$

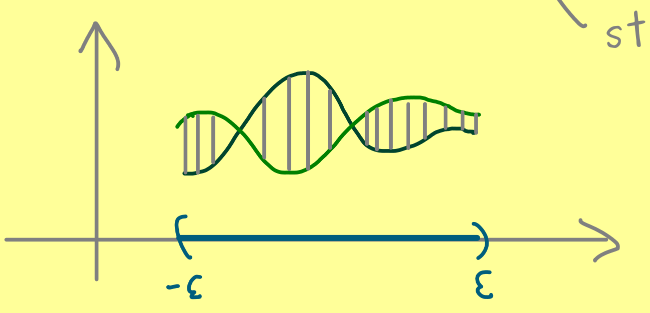
Now we know: $\alpha: \mathbb{R} \to \mathbb{R}^n$ is a solution of $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$

$\iff \Phi(\alpha) = \alpha$ (fixed point equation)

For (1): $X = \left\{ \alpha : (-\varepsilon, \varepsilon) \to \tilde{U} \subseteq \mathbb{R}^n \text{ in the domain of } v \text{ with property } (*) \text{ (see below)} \;\middle|\; \alpha \text{ continuous + bounded},\ \alpha(0) = x_0 \right\}$

with metric: $d(\alpha, \beta) := \sup_{t \in (-\varepsilon,\varepsilon)} \| \alpha(t) - \beta(t) \|_{\mathbb{R}^n}$

standard norm



Fact: (X, d) is a complete metric space.

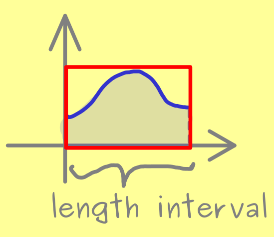
For (2): $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$ gives a map $\Phi : X \to X$

$$d(\Phi(\alpha), \Phi(\beta)) = \sup_{t \in (-\varepsilon,\varepsilon)} \| \Phi(\alpha)(t) - \Phi(\beta)(t) \|_{\mathbb{R}^n}$$

$$= \sup_{t \in (-\varepsilon,\varepsilon)} \left\| \int_0^t \big(v(\alpha(s)) - v(\beta(s)) \big)\, ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\leq \sup_{t \in (-\varepsilon,\varepsilon)} \int_0^t \| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}\, ds$$



$$\leq \sup_{t \in (-\varepsilon,\varepsilon)} \underbrace{\text{length}([0,t])}_{|t| \leq \varepsilon} \cdot \underbrace{\sup_{s \in [0,t]} \| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}}_{\leq \sup_{s \in (-\varepsilon,\varepsilon)} \cdots}$$

$$\leq \varepsilon \cdot \sup_{s \in (-\varepsilon,\varepsilon)} \underbrace{\| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}}_{\leq L\, \| \alpha(s) - \beta(s) \|_{\mathbb{R}^n}}$$

(*) needed

$$\leq \underbrace{\varepsilon \cdot L}_{< 1} \cdot d(\alpha, \beta) \quad \text{contraction}$$

for ε small enough

Picard-Lindelöf theorem

$v: U \to \mathbb{R}^n$ loc. Lipschitz continuous, $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha: (-\varepsilon, \varepsilon) \to U$

for the initial value problem $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$.

Definition of $\tilde{U}$ with property (*)

v being locally Lipschitz continuous at x_0 means:

$$\exists_{\delta > 0} \quad \exists_{L \geq 0} \quad \forall_{y, z \in B_\delta(x)} : \quad \| v(y) - v(z) \| \leq L \cdot \| y - z \|$$

$\alpha(s)$ $\beta(s)$

So we need $\alpha(s), \beta(s) \in B_\delta(x_0)$ for all $s \in (-\varepsilon, \varepsilon)$.

Hence: $\tilde{U} := B_\delta(x_0)$ (not a problem for the solution since we choose ε as small as we want)