The Bright Side of Mathematics

Ordinary Differential Equations – Part 13

Picard-Lindelöf theorem

$v: U \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous , $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U$

for the initial value problem $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$.

Picard iteration:

Iteration from the Banach fixed-point theorem $\Phi^n(\tilde{\alpha}) \xrightarrow{n \to \infty} \alpha$

$$\boxed{\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t v(\tilde{\alpha}(s))\, ds}$$

Example: initial value problem: $\begin{array}{l} \dot{x} = x \\ x(0) = 1 \end{array}$

start with $\tilde{\alpha} : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$, $\tilde{\alpha}(t) = 1$

first step: $\Phi(\tilde{\alpha})(t) = 1 + \int_0^t \tilde{\alpha}(s)\, ds = 1 + t$

second step: $\Phi^2(\tilde{\alpha})(t) = 1 + \int_0^t (1 + s)\, ds = 1 + t + \frac{1}{2} t^2$

nth step: $\Phi^n(\tilde{\alpha})(t) = 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \dots + \frac{1}{n!} t^n$

$\Big\downarrow\ n \to \infty$ (pointwise limit) (also uniform limit)

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} = \exp(t)$$