The Bright Side of Mathematics

Ordinary Differential Equations - Part 17

In part 12: Picard-Lindelöf theorem for initial value problem

$$\boxed{\begin{array}{l}\dot{x} = v(x) \\ x(0) = x_0\end{array}}$$

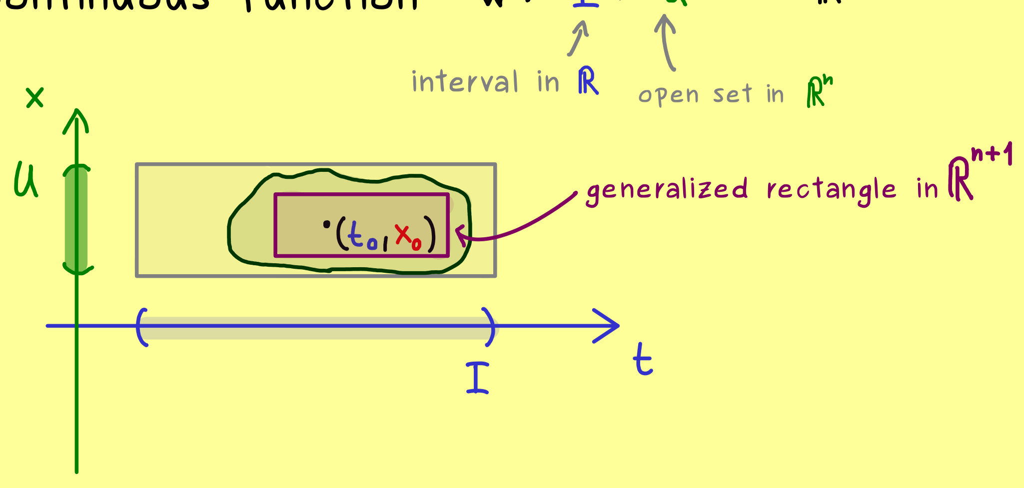
locally Lipschitz continuous

$\Longrightarrow$ there is a unique solution

Now: $\boxed{\begin{array}{l}\dot{x} = w(t,x) \\ x(t_0) = x_0\end{array}}$ initial value problem

continuous function $w: I \times U \to \mathbb{R}^n$

interval in $\mathbb{R}$ open set in $\mathbb{R}^n$



Picard-Lindelöf theorem (for non-autonomous systems)

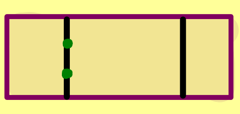
Assume $w: I \times U \to \mathbb{R}^n$ satisfies: $\forall K \subseteq I \times U$ compact $\exists L_K > 0 \ \forall (t,x),(t,y) \in K$:

interval in $\mathbb{R}$ open set in $\mathbb{R}^n$

continuous!

$$\| w(t,x) - w(t,y) \| \leq L_K \cdot \| x - y \|$$

standard norm in $\mathbb{R}^n$



Then: For $x_0 \in U$, there is $\varepsilon > 0$ and a unique solution $\alpha: (t_0-\varepsilon, t_0+\varepsilon) \to U$

for the initial value problem $\boxed{\begin{array}{l}\dot{x} = w(t,x) \\ x(t_0) = x_0\end{array}}$.

Proof: Same as in part 12 with $\Phi(\alpha)(t) = x_0 + \int_{t_0}^{t} w(s, \alpha(s))\, ds$

and Banach fixed-point theorem.

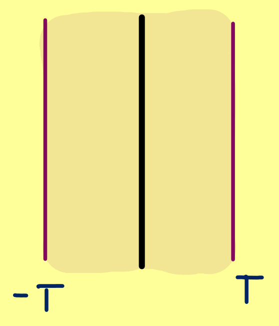
Picard-Lindelöf theorem (special version)

Assume $w: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies: for each $T > 0$:

$$\exists L_T > 0 \ \forall t \in [-T, T] \ \forall x, y \in \mathbb{R}^n: \ \| w(t,x) - w(t,y) \| \leq L_T \cdot \| x - y \|$$

Then there is a unique global solution $\alpha: \mathbb{R} \to \mathbb{R}^n$

for the initial value problem $\boxed{\begin{array}{l}\dot{x} = w(t,x) \\ x(t_0) = x_0\end{array}}$.



Proof: Set $t_0 = 0$. Complete metric space $X = C([-T,T], \mathbb{R}^n)$

with metric $d(\alpha, \beta) := \sup_{t \in [-T,T]} e^{-2L_T|t|} \cdot \| \alpha(t) - \beta(t) \|_{\mathbb{R}^n}$

standard norm

$$\Phi(\alpha)(t) = x_0 + \int_0^t w(s, \alpha(s))\, ds$$

$$d(\Phi(\alpha), \Phi(\beta)) = \sup_{t \in [-T,T]} e^{-2L_T|t|} \| \Phi(\alpha)(t) - \Phi(\beta)(t) \|_{\mathbb{R}^n}$$

$$= \sup_{t \in [-T,T]} e^{-2L_T|t|} \left\| \int_0^t \big(w(s, \alpha(s)) - w(s, \beta(s)) \big) ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\leq \sup_{t \in [-T,T]} e^{-2L_T|t|} \left| \int_0^t \underbrace{\| w(s, \alpha(s)) - w(s, \beta(s)) \|_{\mathbb{R}^n}}_{\leq L_T \cdot \| \alpha(s) - \beta(s) \|_{\mathbb{R}^n}} ds \right|$$

$$\leq \sup_{t \in [-T,T]} e^{-2L_T|t|} \left| \int_0^t L_T \cdot e^{2L_T|s|} \underbrace{e^{-2L_T|s|} \| \alpha(s) - \beta(s) \|_{\mathbb{R}^n}}_{\leq d(\alpha,\beta)} ds \right|$$

$$\leq \sup_{t \in [-T,T]} e^{-2L_T|t|} L_T \cdot d(\alpha, \beta) \underbrace{\left| \int_0^t e^{2L_T|s|} ds \right|}_{\frac{1}{2L_T}\left(e^{2L_T|t|} - 1\right)}$$

$$\leq \frac{1}{2} d(\alpha, \beta) \underbrace{\sup_{t \in [-T,T]} \left(1 - e^{-2L_T|t|}\right)}_{\leq 1}$$

$$\leq \frac{1}{2} d(\alpha, \beta)$$

$\Longrightarrow$ $\Phi: X \to X$ is a contraction $\overset{\text{Banach fixed-point theorem}}{\Longrightarrow}$ unique solution $\alpha: [-T,T] \to \mathbb{R}^n$

for all $T > 0$

$\Longrightarrow$ global solution $\alpha: \mathbb{R} \to \mathbb{R}^n$ $\square$