Ordinary Differential Equations - Part 20

System of linear differential equations (homogeneous + autonomous)

$$(\mathrm{IVP}^{0}_{x_0}) \quad \boxed{\begin{aligned} \dot{x} &= A x \\ x(0) &= x_0 \end{aligned}} \quad , \quad \begin{aligned} A &\in \mathbb{R}^{n\times n} \\ x_0 &\in \mathbb{R}^n \end{aligned}$$

$\hookrightarrow$ $\underbrace{\text{Picard iteration (see part 13)}}$

start with a guess $\tilde{\alpha}: \mathbb{R} \longrightarrow \mathbb{R}^n$

$$\boxed{\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t A\,\tilde{\alpha}(s)\, ds} \rightsquigarrow \Phi^n(\tilde{\alpha}) \xrightarrow{n\to\infty} \alpha$$

solution of $(\mathrm{IVP}^{0}_{x_0})$

Picard iteration: guess: $\tilde{\alpha}: \mathbb{R} \longrightarrow \mathbb{R}^n$, $\tilde{\alpha}(t) = x_0$

1st step: $\Phi(\tilde{\alpha})(t) = x_0 + \int_0^t A x_0 \, ds = (\mathbb{1} + tA)x_0$

2nd step: $\Phi^2(\tilde{\alpha})(t) = x_0 + \int_0^t A\big((\mathbb{1} + tA)x_0\big)\, ds$

$$= x_0 + tAx_0 + \tfrac{1}{2}t^2A^2x_0 = \left(\mathbb{1} + tA + \tfrac{1}{2}t^2A^2\right)x_0$$

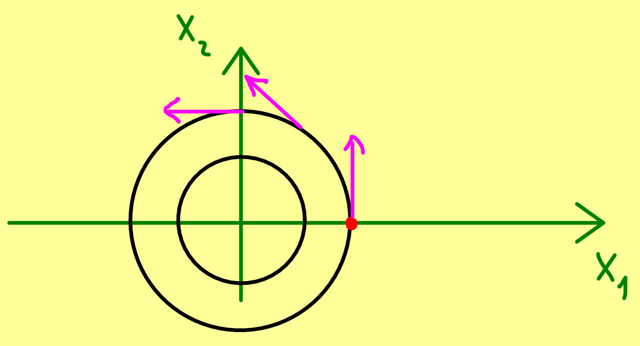
nth step: $\Phi^n(\tilde{\alpha})(t) = \left(\mathbb{1} + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{6}t^3A^3 + \cdots + \tfrac{1}{n!}t^nA^n\right)x_0$

$\xrightarrow{n\to\infty}$ solution of $(\mathrm{IVP}^{0}_{x_0})$ $\quad \alpha(t) = \underbrace{\sum_{k=0}^{\infty} \frac{(t\cdot A)^k}{k!}}_{=:\ \exp(t\cdot A) \ = \ e^{tA}} x_0$

matrix exponential

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\exp(t\cdot A) = \Big(\mathbb{1} + t\underbrace{A}_{=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}} + \tfrac{1}{2}t^2\underbrace{A^2}_{=\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}} + \tfrac{1}{6}t^3\underbrace{A^3}_{=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}} + \tfrac{1}{4!}t^4\underbrace{A^4}_{=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \mathbb{1}} + \cdots\Big)$$

$$= \begin{pmatrix} \underbrace{1 - \tfrac{1}{2}t^2 + \tfrac{1}{4!}t^4 \pm \cdots}_{=\cos(t)} & -\sin(t) \\ \underbrace{0 + t - \tfrac{1}{6}t^3 + \tfrac{1}{5!}t^5 \pm \cdots}_{=\sin(t)} & \cos(t) \end{pmatrix}$$

solution of $(\mathrm{IVP}^{0}_{x_0})$ with $x_0 = \begin{pmatrix} c \\ 0 \end{pmatrix}$:

$$\alpha(t) = \exp(t\cdot A)\, x_0 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = c \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$\Longrightarrow$ all orbits are circles!