The Bright Side of Mathematics

Probability Theory – Part 5

Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

sample space, σ-algebra, probability measure

$\mathcal{A} \subseteq \mathcal{P}(\Omega)$, $\mathbb{P}: \mathcal{A} \to [0,1]$

$\rightsquigarrow (\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$, $n \in \{1,2,\dots\}$

Example: first throw a die, then throw a point into the interval $[-1, 1]$

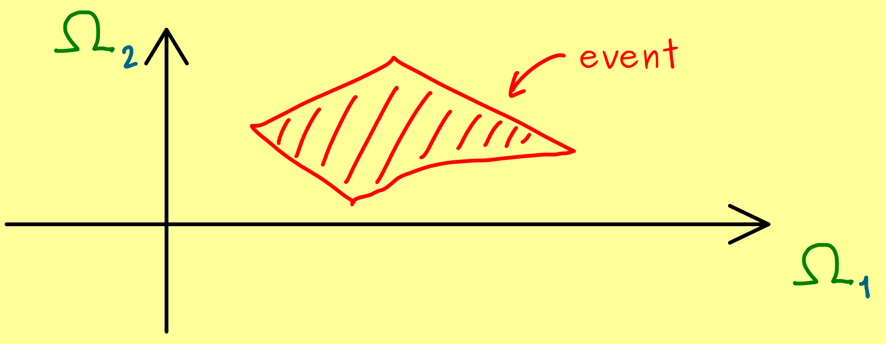
possible outcome: $(3, \frac{1}{4})$ probability?

First probability space: $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$

$\Omega_1 = \{1,\dots,6\}$, $\mathcal{A}_1 = \mathcal{P}(\Omega)$, $\mathbb{P}_1(A) = \sum_{k \in A} \frac{1}{6}$

Second probability space: $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$

$\Omega_2 = [-1, 1]$, $\mathcal{A}_2 = \mathcal{B}(\Omega)$, $\mathbb{P}_2(A) = \int_A \frac{1}{2}\, dx$



new probability space

$(\Omega_1 \times \Omega_2, \sigma(\mathcal{A}_1 \times \mathcal{A}_2), \mathbb{P})$

product σ-algebra, product measure

$\mathbb{P}$ satisfies for $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$

$$\mathbb{P}(A_1 \times A_2) = \mathbb{P}_1(A_1) \cdot \mathbb{P}_2(A_2)$$

$$\mathbb{P}(\{2,3\} \times [-1,0]) = \mathbb{P}_1(\{2,3\}) \cdot \mathbb{P}_2([-1,0]) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Definition: Probability spaces: $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$, $n \in \{1,2,\dots\}$

Product space: $(\Omega, \mathcal{A}, \mathbb{P})$ defined by:

- $\Omega = \Omega_1 \times \Omega_2 \times \cdots = \prod_{j \in \mathbb{N}} \Omega_j$ (elements: $(\omega_1, \omega_2, \omega_3, \dots)$)
- $\mathcal{A} = \sigma(\text{"cylinder sets"})$ product σ-algebra
  $\Omega_1 \times \Omega_2 \times A_3 \times \Omega_4 \times \cdots$
  $A_1 \times \Omega_2 \times \Omega_3 \times \Omega_4 \times \cdots$
- $\mathbb{P}$ product measure

$$\mathbb{P}(A_1 \times A_2 \times \cdots \times A_m \times \Omega_{m+1} \times \Omega_{m+2} \times \cdots) = \mathbb{P}_1(A_1) \cdot \mathbb{P}_2(A_2) \cdot \ldots \cdot \mathbb{P}_m(A_m)$$

Example: throw a die infinitely many times: $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$

$\Omega_0 = \{1,\dots,6\}$, $\mathcal{A}_0 = \mathcal{P}(\Omega)$, $\mathbb{P}_0(A) = \sum_{k \in A} \frac{1}{6}$

Product space: $\Omega = \Omega_0 \times \Omega_0 \times \cdots$, $\mathcal{A}$ = product σ-algebra, $\mathbb{P}$ product measure

$A \in \mathcal{A}$ event: "At the 100th throw, we get a six for the first time"

$$A = \underbrace{\{6\}^c \times \{6\}^c \times \cdots \times \{6\}^c}_{99 \text{ times}} \times \{6\} \times \Omega_0 \times \Omega_0 \times \cdots$$

$$\mathbb{P}(A) = \mathbb{P}_0(\{6\}^c) \cdot \ldots \cdot \mathbb{P}_0(\{6\}^c) \cdot \mathbb{P}_0(\{6\}) = \mathbb{P}_0(\{6\}^c)^{99} \cdot \mathbb{P}_0(\{6\}) = \left(\frac{5}{6}\right)^{99} \cdot \frac{1}{6}$$