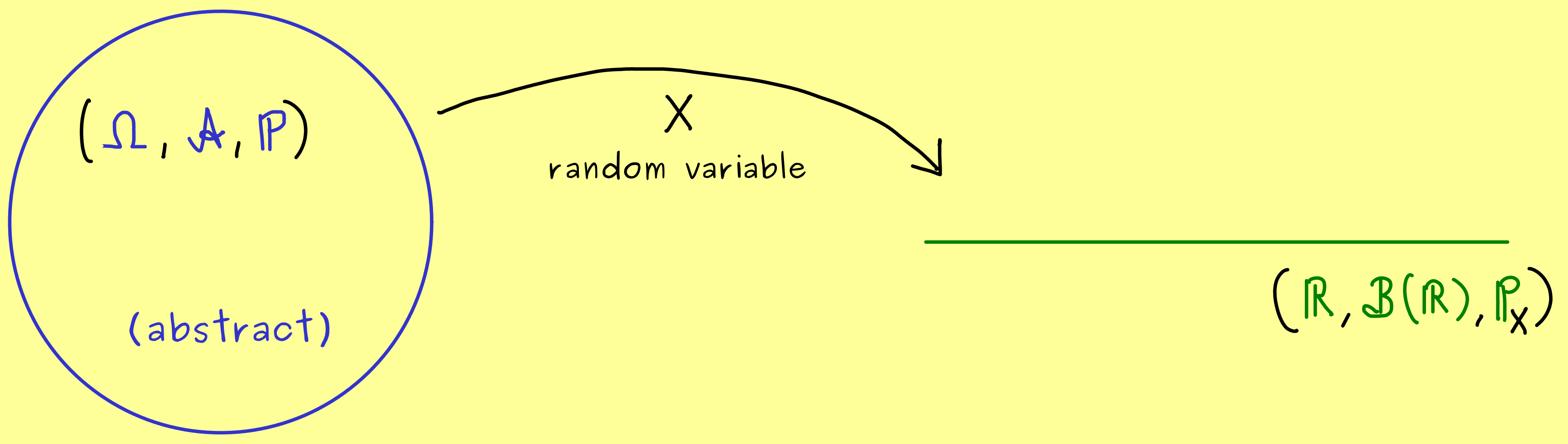
The Bright Side of Mathematics

Probability Theory – Part 11

$(\Omega, \mathcal{A}), (\underbrace{\tilde{\Omega}}_{= \mathbb{R}}, \underbrace{\tilde{\mathcal{A}}}_{= \mathcal{B}(\mathbb{R})})$ event spaces, $X : \Omega \longrightarrow \tilde{\Omega}$



Definition: Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $X : \Omega \longrightarrow \mathbb{R}$ be a random variable.
(with Borel sigma algebra)

Then $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \longrightarrow [0,1]$ defined by

$$\mathbb{P}_X(B) := \mathbb{P}(X^{-1}(B)) = \mathbb{P}(X \in B)$$

is called probability distribution of X.

Proposition: $\mathbb{P}_X$ is a probability measure.

Proof:
$\mathbb{P}$ is a probability measure

$$X^{-1}(\mathbb{R}) = \Omega \quad \Rightarrow \quad \mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1$$

$$X^{-1}(\emptyset) = \emptyset \quad \Rightarrow \quad \mathbb{P}_X(\emptyset) = \mathbb{P}(X^{-1}(\emptyset)) = \mathbb{P}(\emptyset) = 0$$

For σ-additivity: Choose $B_1, B_2, B_3, \dots \in \mathcal{B}(\mathbb{R})$ pairwise disjoint.

Then: $i \neq j \Rightarrow X^{-1}(B_i) \cap X^{-1}(B_j) = X^{-1}(\underbrace{B_i \cap B_j}_{= \emptyset}) = \emptyset$

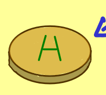
So: $X^{-1}(B_1), X^{-1}(B_2), X^{-1}(B_3) \dots \in \mathcal{A}$ pairwise disjoint.

And: $\mathbb{P}_X\left(\bigcup_{j=1}^{\infty} B_j\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right)\right) = \mathbb{P}\left(\bigcup_{j=1}^{\infty} X^{-1}(B_j)\right)$

$\mathbb{P}$ is a probability measure

$$= \sum_{j=1}^{\infty} \mathbb{P}(X^{-1}(B_j)) = \sum_{j=1}^{\infty} \mathbb{P}_X(B_j) \qquad \square$$

Notation: If $\tilde{\mathbb{P}}$ probability measure and $\mathbb{P}_X = \tilde{\mathbb{P}}$, then $X \sim \tilde{\mathbb{P}}$.

Example:  p n tosses of the same coin $(\underbrace{\Omega}_{= \{0,1\}^n}, \underbrace{\mathcal{A}}_{= \mathcal{P}(\Omega)}, \mathbb{P})$

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$\mathbb{P}(\{\omega\}) = p^{\#1s} \cdot (1-p)^{\#0s}$

$X : \Omega \longrightarrow \mathbb{R}$

$X(\omega) :=$ number of 1s in ω $\xRightarrow{\text{part 4}}$ $X \sim \mathrm{Bin}(n, p)$