The Bright Side of Mathematics

Probability Theory – Part 18

Properties of variance and standard deviation:

Let X, Y be independent random variables where $\mathbb{E}(X^2)$ and $\mathbb{E}(Y^2)$ exist.

Then:
(a) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

(b) $\mathrm{Var}(\lambda \cdot X) = \lambda^2 \cdot \mathrm{Var}(X)$ for every $\lambda \in \mathbb{R}$

(c) $\sigma(\lambda \cdot X) = |\lambda| \cdot \sigma(X)$ for every $\lambda \in \mathbb{R}$

Proof: (a)

$$\mathrm{Var}(X+Y) = \mathbb{E}\big((X+Y)^2\big) - \mathbb{E}(X+Y)^2$$
$$= \mathbb{E}(X^2 + 2XY + Y^2) - \big(\mathbb{E}(X) + \mathbb{E}(Y)\big)^2$$
$$= \mathbb{E}(X^2) + 2\,\mathbb{E}(XY) + \mathbb{E}(Y^2) - \mathbb{E}(X)^2 - 2\,\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2$$
$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\cdot\big(\underbrace{\mathbb{E}(XY)}_{= \mathbb{E}(X)\mathbb{E}(Y)} - \mathbb{E}(X)\mathbb{E}(Y)\big)$$

independence

(b)

$$\mathrm{Var}(\lambda \cdot X) = \mathbb{E}\big((\lambda \cdot X)^2\big) - \mathbb{E}(\lambda \cdot X)^2$$
$$= \lambda^2\, \mathbb{E}\big((X)^2\big) - \lambda^2\, \mathbb{E}(X)^2 = \lambda^2 \cdot \big(\mathbb{E}(X^2) - \mathbb{E}(X)^2\big)$$
$$= \lambda^2 \cdot \mathrm{Var}(X)$$

(c)

$$\sigma(\lambda \cdot X) = \sqrt{\mathrm{Var}(\lambda \cdot X)} \overset{(b)}{=} |\lambda| \cdot \sigma(X)$$