

QR-decomposition

Let A be square matrix + invertible

$$A = Q \cdot R \quad \left(\begin{array}{c} \text{diagonal} \\ \text{below} \\ 0 \end{array} \right)$$

$Q^T = Q^{-1}$ ← orthogonal / unitary matrix $\rightsquigarrow Q^* = Q^{-1}$
 (columns of Q form ONB)

related to Gram-Schmidt process

$$A = \left(\begin{array}{c|c|c} a_1 & a_2 & a_3 \\ \hline \hline \hline \end{array} \right) \in \mathbb{R}^{3 \times 3} \rightsquigarrow Q = \left(\begin{array}{c|c|c} q_1 & q_2 & q_3 \\ \hline \hline \hline \end{array} \right)$$

① $q_1 := \frac{a_1}{\|a_1\|}$ (hence: $\|q_1\| = 1$)

$$R = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & 0 & \square \end{pmatrix}$$

② $a_2^\perp = a_2 - \langle a_2, q_1 \rangle q_1$

$$q_2 := \frac{a_2^\perp}{\|a_2^\perp\|}$$

③ $a_3^\perp = a_3 - \langle a_3, q_1 \rangle q_1 - \langle a_3, q_2 \rangle q_2$

$$q_3 := \frac{a_3^\perp}{\|a_3^\perp\|}$$

Example: $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix}$

① $q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② $a_2^\perp = a_2 - \langle a_2, q_1 \rangle q_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle}_{= \frac{1}{\sqrt{2}}(3-1) = \frac{2}{\sqrt{2}} = \sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$q_2 = \frac{1}{\underbrace{\left(\sqrt{2^2 + (-2)^2}\right)}_{= 2\sqrt{2}}} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

Another example: $A = \begin{pmatrix} 2 & -2 & -12 \\ 4 & 2 & -18 \\ -4 & -8 & 30 \end{pmatrix}$