QR-decomposition

Let A be square matrix + invertible

$$A = Q \cdot R \leftarrow \begin{pmatrix} & \ddots \\ 0 & \end{pmatrix}$$

[related to Gram-Schmidt process]

$Q^T = Q^{-1}$ ← orthogonal / unitary matrix ⇝ $Q^* = Q^{-1}$

(columns of Q form ONB)

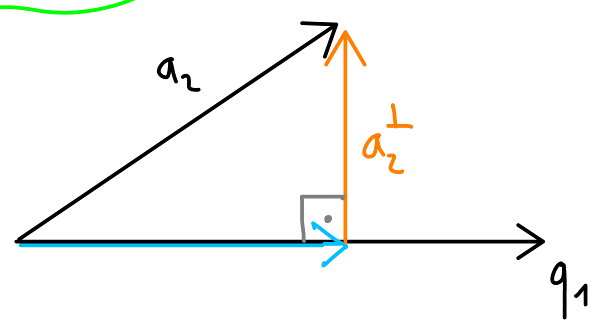
$$A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} \in \mathbb{R}^{3 \times 3} \rightsquigarrow Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}$$

$$\rightsquigarrow R = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & 0 & \square \end{pmatrix}$$

① $q_1 := \dfrac{a_1}{\boxed{\|a_1\|}}$ (hence: $\|q_1\| = 1$)

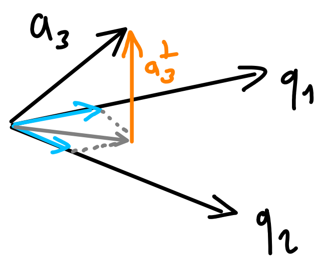
② $a_2^\perp = a_2 - \boxed{\langle a_2, q_1 \rangle} q_1$

$q_2 := \dfrac{a_2^\perp}{\boxed{\|a_2^\perp\|}}$



③ $a_3^\perp = a_3 - \boxed{\langle a_3, q_1 \rangle} q_1 - \boxed{\langle a_3, q_2 \rangle} q_2$

$q_3 := \dfrac{a_3^\perp}{\boxed{\|a_3^\perp\|}}$



Example: $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix}$

1. $q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2. $a_2^\perp = a_2 - \langle a_2, q_1 \rangle q_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \underbrace{\left\langle \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle}_{= \frac{1}{\sqrt{2}}(3-1) = \frac{2}{\sqrt{2}} = \sqrt{2}} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$q_2 = \frac{1}{\underbrace{\left(\sqrt{2^2 + (-2)^2}\right)}_{= 2\sqrt{2}}} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Another example:

$$A = \begin{pmatrix} 2 & -2 & -12 \\ 4 & 2 & -18 \\ -4 & -8 & 30 \end{pmatrix}$$