Introduction

Definition 1. Real numbers

The real numbers are a non-empty set $\mathbb{R}$ *together with the operations* $+ : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and an ordering relation* $\leq : \mathbb{R} \times \mathbb{R} \to \{\mathit{True}, \mathit{False}\}$ *that fulfil the following rules*

- *(A) Addition*
  - *(A1) associative:* $x + (y + z) = (x + y) + z$
  - *(A2) neutral element: There is a (unique) element* 0 *with* $x + 0 = x$ *for all* x.
  - *(A3) inverse element: For all* x *there is a (unique)* y *with* $x + y = 0$. *We write for this element simply* $-x$.
  - *(A4) commutative:* $x + y = y + x$
- *(M) Multiplication*
  - *(M1) associative:* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  - *(M2) neutral element: There is a (unique) element* $1 \neq 0$ *with* $x \cdot 1 = x$ *for all* x.
  - *(M3) inverse element: For all* $x \neq 0$ *there is a (unique)* y *with* $x \cdot y = 1$. *We write for this element simply* x^{-1}.
  - *(M4) commutative:* $x \cdot y = y \cdot x$
- *(D) Distributivity:* $x \cdot (y + z) = x \cdot y + x \cdot z$.
- *(O) Ordering*
  - *(O1)* $x \leq x$ *is true for all* x.
  - *(O2) If* $x \leq y$ *and* $y \leq x$, *then* $x = y$.
  - *(O3) transitive:* $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.
  - *(O4) For all* $x, y \in X$, *we have either* $x \leq y$ *or* $y \leq x$.
  - *(O5)* $x \leq y$ *implies* $x + z \leq y + z$ *for all* z.
  - *(O6)* $x \leq y$ *implies* $x \cdot z \leq y \cdot z$ *for all* $z \geq 0$.
  - *(O7)* $x > 0$ *and* $\varepsilon > 0$ *implies* $x < \varepsilon + \cdots + \varepsilon$ *for sufficiently many summands.*
- *(C) Let* $X, Y \subset \mathbb{R}$ *be two non-empty subsets with the property* $x \leq y$ *for all* $x \in X$ *and* $y \in Y$. *Then there is a* $c \in \mathbb{R}$ *with* $x \leq c \leq y$ *for all* $x \in X$ *and* $y \in Y$.

Remark 2.

In the video, we described the completeness axiom with the help of sequences. Then it sounds like:

Completeness: Every sequence $(a_n)_{n\in\mathbb{N}}$ with the property *[For all* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *with* $|a_n - a_m| < \varepsilon$ *for all* $n, m > N$*]* has a limit.

Later, we will show the equivalence of both descriptions.

Definition 3. Absolute value for real numbers

*The **absolute value** of a number $x \in \mathbb{R}$ is defined by*

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Exercise 4.

Use the axioms to show:

$(1): \ 0 \cdot x = 0$

$(2): \ -x = (-1) \cdot x$

$(3): \ (-1) \cdot (-1) = 1$

$(4): \ 1 > 0$

(1): $0 \cdot x \overset{(A2)}{=} (0+0)x \overset{(D)}{=} 0x + 0x \overset{(A2)}{\Longrightarrow} 0x = 0$

(2): $-x \overset{(*)}{=} 0x + (-x) \overset{(A3)}{=} (1+(-1))x + (-x) \overset{(D)}{=} x + (-1)x + (-x) \overset{(A2-4)}{=} (-1)x$

(3): $x(-1)(-1) \overset{(**)}{=} -(-x) \overset{(A3)}{=} x \overset{(M2)}{\Longrightarrow} (-1) \cdot (-1) = 1$

(4): Try for yourself.

Exercise 5. Learn to sketch

Sketch the following subsets of $\mathbb{R}$:

(a) $A := \{x \in \mathbb{R} \mid x < x^2\}$

(b) $B := \left\{x \in \mathbb{R} \mid x > 0 \wedge \frac{10}{x} - 3 \leq \frac{4}{x} + 1\right\}$

Exercise 6. Sketch in two dimensions

Sketch the following subsets of $\mathbb{R}^2$:

(a) $M_1 := \{(x,y) \in \mathbb{R}^2 \mid (x-3)^2 + (y+4)^2 \leq 9\}$

(b) $M_2 := \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 1\}$

Exercise 7. Use the axioms!

Prove for $x, y, z \in \mathbb{R}$ the following statements by using the axioms of the real numbers:

(a) $x \leq y \wedge z \leq 0 \ \Rightarrow \ xz \geq yz$

(b) $\big||x| - |y|\big| \leq |x - y|$

(c) $|xy| \leq \frac{1}{2}(x^2 + y^2)$

(d) $x, y \geq 0 \ \Rightarrow \ xy \leq \frac{1}{4}(x+y)^2$