Sandwich Theorem

Theorem 1. Monotonicity of limits

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent real sequences with

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

Further, assume that for all $n \in \mathbb{N}$ holds $a_n \leq b_n$. Then the following holds true:

(i) $a \leq b$;

(ii) Sandwich-Theorem*: If $a = b$ and $(c_n)_{n\in\mathbb{N}}$ is another sequence with $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, then $(c_n)_{n\in\mathbb{N}}$ is convergent with*

$$\lim_{n\to\infty} c_n = a = b.$$

Proof. (i) Consider the sequence of differences between b_n and a_n, i.e., $(b_n - a_n)_{n\in\mathbb{N}}$. By the calculation rules for converging sequences, it suffices to show that

$$b - a = \lim_{n\to\infty}(b_n - a_n) \geq 0.$$

Assume the converse statement, i.e., $b - a < 0$. Then, we have that both numbers $a - b$ and $b_n - a_n$ are positive and thus

$$|a - b - (a_n - b_n)| = a - b + (b_n - a_n) > a - b.$$

In particular, there exists no $n \in \mathbb{N}$ such that $|a-b-(a_n-b_n)| < \varepsilon$ for $\varepsilon = a-b > 0$. This is a contradiction to $\lim_{n\to\infty}(b_n - a_n) = b - a$.

(ii) Again consider the sequence $(b_n - a_n)_{n\in\mathbb{N}}$ which is tending to zero according to the formulae for converging sequences. Further, consider the sequence $(c_n - a_n)_{n\in\mathbb{N}}$. Then we have for all $n \in \mathbb{N}$ that $0 \leq c_n - a_n \leq b_n - a_n$. Let $\varepsilon > 0$. Since $b_n - a_n$ is tending to zero, there exists some N such that for all $n \geq N$ holds $|b_n - a_n - 0| < \varepsilon$. Due to $0 \leq c_n - a_n \leq b_n - a_n$, we can conclude that for $n \geq N$ holds

$$|c_n - a_n - 0| = c_n - a_n \leq b_n - a_n = |b_n - a_n - 0| < \varepsilon.$$

This implies that $(c_n - a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty}(c_n - a_n) = 0$. Hence $a = 0 + a = \lim_{n\to\infty}(c_n - a_n) + \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$.

□

Remark 2.

Since the modification of finitely many sequence elements does not change the limits (take a closer look at the definition of convergence, the statements of Theorem 1 can be slightly generalised by only claiming that there exists some n_0 such that for all $n \geq n_0$ holds $a_n \leq b_n$ (resp. for all $n \geq n_0$ holds $a_n \leq c_n \leq b_n$ in (ii)). In the proof of (i), one has to replace the words "there exists no $n \in \mathbb{N}$ such that" by "there exists no $n \geq n_0$ such that" and in the proof of (ii) the number N has to be replaced by

$\max\{N, n_0\}$.

Attention!

From the fact that we have the strict inequality $a_n < b_n$, we cannot conclude that the limits satisfy $a < b$. To see this, consider the sequences $(a_n)_{n\in\mathbb{N}} = (0, 0, 0, \dots)$ and $(b_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$. In this case, we have $a = b = 0$ though the strict inequality $a_n = 0 < \frac{1}{n} = b_n$ holds true for all $n \in \mathbb{N}$.

Example 3. (a) Consider $(\frac{1}{n^k})_{n\in\mathbb{N}}$ for some $k \in \mathbb{N}$. We state two alternative ways to show that this sequence tends to zero. The first possibility is, of course, an argumentation as in statement (b) of the remarks on limit theorems. The second way to treat this problem is by making use of the inequality

$$\frac{1}{n} \geq \frac{1}{n^k} > 0.$$

Since we know from example q^n, the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ tends to zero, statement (ii) of Theorem 1 directly leads to the fact that $(\frac{1}{n^k})_{n\in\mathbb{N}}$ also tends to zero.

(b) Consider $(a_n)_{n\in\mathbb{N}}$ with

$$a_n = \frac{2n^2 + 5n - 1}{-5n^2 + n + 1}.$$

Rewriting

$$a_n = \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}},$$

and using that both $(\frac{1}{n})_{n\in\mathbb{N}}$ and $(\frac{1}{n^2})_{n\in\mathbb{N}}$ tend to zero, we can apply the limit theorems to obtain that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2 + 5n - 1}{-5n^2 + n + 1} = \lim_{n\to\infty} \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}} = -\frac{2}{5}.$$

(c) Consider $(a_n)_{n\in\mathbb{N}}$ with $a_n = \sqrt{n^2+1} - n$. At first glance, none of the so far presented results seem to help to analyse convergence of this sequence. However, we can compute

$$\begin{aligned} a_n &= \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} \\ &= \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n} < \frac{1}{n}. \end{aligned}$$

By Theorem 1, we now get that $\lim_{n\to\infty} a_n = 0$.

Exercise 4.

The following sequences are convergent. Find their limits and justify your steps by clearly mentioning the properties/previously known results that you have used.

(a) $a_n := \frac{2}{n} - \frac{6}{n^2}$

(b) $b_n := \frac{n^4 - 8n^3 + 12n}{2n^5 + 8n^2 - 4}$

(c) $c_n := \frac{3^n}{n!}$

(d) $d_n := n^k q^n$ *with* $q \in \mathbb{R}$, $|q| < 1$ *and* $k \in \mathbb{N}_0$

(e) $e_n := \sqrt{n}\left(\sqrt{n+2} - \sqrt{n}\right)$,

(f) $f_n := \sqrt[n]{n^k}$ *with* $k \in \mathbb{N}_0$,

(g) $g_n := \left(1 - \frac{1}{n^2}\right)^n$,

(h) $h_{n+1} := \frac{h_n + 3}{4}$, $h_1 := 0$.