Supremum and Infimum

Definition 1. Supremum and Infimum

Let $M \subset \mathbb{R}$ be a set.

(a) A real number s is called the supremum *of M if:*

- $x \leq s$ *for all* $x \in M$,
- *for all* $\varepsilon > 0$ *there is an* $x \in M$ *with* $s - \varepsilon < x$.

In this case we write $s = \sup M$.

(b) A real number l is called the infimum *of M if:*

- $x \geq l$ *for all* $x \in M$,
- *for all* $\varepsilon > 0$ *there is an* $x \in M$ *with* $l + \varepsilon > x$.

In this case we write $l = \inf M$.

(c) We further define

- $\sup M = \infty$ *if M is not bounded from above;*
- $\inf M = -\infty$ *if M is not bounded from below;*
- $\sup \emptyset = -\infty$;
- $\inf \emptyset = \infty$.

To remember: Sup and Inf

The infimum is the greatest lower bound and the supremum is the lowest upper bound.

Example 2. (a) $\sup[0, 1] = 1, \quad \inf[0, 1] = 0;$

(b) $\sup(0, 1) = 1, \quad \inf(0, 1) = 0;$

(c) $\sup\{\frac{1}{n} : n \in \mathbb{N}\} = 1, \qquad \inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0;$

(d) $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sqrt{2}, \qquad \inf\{x \in \mathbb{Q} : x^2 < 2\} = -\sqrt{2};$

Remark 3. Difference between sup and max (resp. inf and min)

In contrast to the maximum, the supremum does not need to belong to the respective set. For instance, we have $1 = \sup(0, 1)$, but $\max(0, 1)$ does not exist. The analogous statement holds true for inf *and* min. *However, we can make the following statement: If $\max M$ ($\min M$) exists, then $\max M = \sup M$ ($\min M = \inf M$).*

Exercise 4.

Determine the supremum and infimum and, in case of existence, the minimum and maximum of the following sets:

(a) $M := \left\{-2 + \frac{6}{4n^3} \mid n \in \mathbb{N}\right\}$,

(b) $A := \{x \in \mathbb{R} \mid |x-7| - |x-4| > -4\}$,

(c) $B := \left\{ \frac{5n}{3^n} \mid n \in \mathbb{N} \right\}$.

(d) $M := \{x \in \mathbb{R} \mid x^2 + 2x < 3\}$.