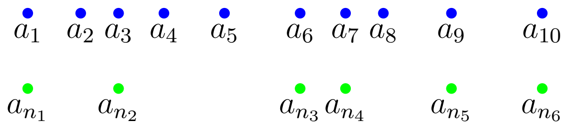
Subsequences and Accumulation Values

Definition 1. Subsequence

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}$. Let $(n_k)_{k\in\mathbb{N}}$ be a strongly monotonically increasing sequence with $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Then $(a_{n_k})_{k\in\mathbb{N}}$ is called a subsequence*.*



Example 2. Consider the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$. Then some subsequences are given by

- $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{k^2})_{k\in\mathbb{N}} = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{2^k})_{k\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots)$;
- $(a_{n_k})_{k\in\mathbb{N}} = (a_{k!})_{k\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots)$.

Theorem 3. Convergence of subsequences

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{F}$ with $\lim_{n\to\infty} a_n = a$. Then all subsequences $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ are also convergent with

$$\lim_{k\to\infty} a_{n_k} = a.$$

Proof: Since $1 \leq n_1 < n_2 < n_3 < \ldots$ and $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$, we have that $n_k \geq k$ for all $k \in \mathbb{N}$. Let $\varepsilon > 0$. By the convergence of $(a_n)_{n\in\mathbb{N}}$, there exists some N such that $|a_k - a| < \varepsilon$ for all $k \geq N$. Due to $n_k \geq k$, we thus also have that $|a_{n_k} - a| < \varepsilon$ for all $k \geq N$. □

Attention!

The existence of a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ does in general <u>not</u> imply the convergence of $(a_n)_{n\in\mathbb{N}}$. For instance, consider $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. Both subsequences

$$\begin{aligned} (a_{2k})_{k\in\mathbb{N}} &= ((-1)^{2k})_{k\in\mathbb{N}} &&= (1, 1, 1, 1, \ldots) \\ (a_{2k+1})_{k\in\mathbb{N}} &= ((-1)^{2k+1})_{k\in\mathbb{N}} &&= (-1, -1, -1, -1, \ldots) \end{aligned}$$

are convergent though $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ is divergent.

However, we can "rescue" this statement by additionally claiming that $(a_n)_{n\in\mathbb{N}}$ is monotonic.

Theorem 4. Subsequences of monotonic sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(a_n)_{n\in\mathbb{N}}$ is monotonic and there exists a convergent *subsequence* $(a_{n_k})_{k\in\mathbb{N}}$*, then* $(a_n)_{n\in\mathbb{N}}$ *is convergent with*

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{n_k}.$$

Proof: Denote $a = \lim_{k\to\infty} a_{n_k}$. We just consider the case where $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing (the remaining part can be done analogously to the argumentations at the end of the proof of Theorem about bounded sequences). Since $(a_{n_k})_{k\in\mathbb{N}}$ is also monotonically increasing, we have that $a = \sup\{a_{n_k} : k \in \mathbb{N}\}$.
Let $\varepsilon > 0$. Due to the convergence and monotonicity of $(a_{n_k})_{k\in\mathbb{N}}$, there exists some $K \in \mathbb{N}$ such that for all $k \geq K$ holds

$$a - \varepsilon < a_{n_k} \leq a.$$

Now assume that $n \geq N = n_K$. Monotonicity then implies that $a - \varepsilon < a_{n_K} \leq a_n \leq a_{n_n} \leq a$. In particular, we have that

$$|a - a_n| = a - a_n < \varepsilon.$$

□

Definition 5. Accumulation value

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$*. Then* $a \in \mathbb{R}$ *is called* accumulation value *if there exists some subsequence* $(a_{n_k})_{k\in\mathbb{N}}$ *with*

$$a = \lim_{k\to\infty} a_{n_k}.$$

Attention! Names

Accumulation values are often called by other names, like accumulation points*,* limits points *or* cluster points*.*

Proposition 6.

$a \in \mathbb{R}$ *is an accumulation value if and only if in every* ε*-neighbourhood of* a*, there are infinitely many elements of the sequence* $(a_n)_{n\in\mathbb{N}}$*.*

Definition 7. Accumulation values $\pm\infty$

A real sequence $(a_n)_{n\in\mathbb{N}}$ *is said to have the* (improper) accumulation value ∞ *if it is not bounded from above. Analogously, we define the* (improper) accumulation value $-\infty$ *if it is not bounded from below.*

Exercise 8.

Determine all accumulation values of the following sequences:

(a) $a_n := \left(1 + \frac{1}{n}\right)^{n+5}$

(b) $b_n := \left(\frac{3}{2}\right)^{(-1)^n \cdot n}$

(c) $c_n := \begin{cases} 0 & n \text{ is odd} \\ \frac{n!}{(n+1)!} & n \text{ is even} \end{cases}$

(d) $d_n := (-3)^n + (1 + (-1)^{3n}) \left(2 + \frac{(-1)^{4n}}{n^2}\right)$