

## Limit Superior and Limit Inferior

### Definition 1. Limit superior - limit inferior

Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence. A number  $a \in \mathbb{R} \cup \{\infty, -\infty\}$  is called

- limit superior of  $(a_n)_{n \in \mathbb{N}}$  if  $a$  is the largest accumulation value of  $(a_n)_{n \in \mathbb{N}}$ . In this case, we write

$$a = \limsup_{n \rightarrow \infty} a_n.$$

- limit inferior of  $(a_n)_{n \in \mathbb{N}}$  if  $a$  is the smallest accumulation value of  $(a_n)_{n \in \mathbb{N}}$ . In this case, we write

$$a = \liminf_{n \rightarrow \infty} a_n.$$

### Remark 2.

Almost needless to say, we define the ordering between infinity and real numbers by  $-\infty < a < \infty$  for all  $a \in \mathbb{R}$ . It can be shown that (in contrast to the limit) the limit superior and limit inferior always exist for any real sequence. This will follow from the subsequent results.

### Lemma 3.

Let  $(a_n)_{n \in \mathbb{R}}$  be a real sequence. Then the following statements hold

$$a) \liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf \{a_k \mid k \geq n\}$$

$$b) \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup \{a_k \mid k \geq n\}$$

c) A sequence is convergent if and only if  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n \notin \{\pm\infty\}$ . In this case holds  $\lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$ .

d) A sequence is divergent to  $\infty$  if and only if  $\liminf_{n \rightarrow \infty} a_n = \infty$ . In this case also holds  $\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \infty$ .

e) A sequence is divergent to  $-\infty$  if and only if  $\limsup_{n \rightarrow \infty} a_n = -\infty$ . In this case also holds  $\lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = -\infty$ .

*Proof:*

- a) If  $(a_n)$  is not bounded from below, then,  $-\infty$  is an accumulation value of  $(a_n)$  which necessarily must be the smallest one. So  $\liminf a_n = -\infty$ . On the other hand, the unboundedness from below of  $(a_n)$  implies  $s_n := \inf\{a_k \mid k \geq n\} = -\infty$  for all  $n \in \mathbb{N}$  and therefore also  $\lim_{n \rightarrow \infty} s_n = -\infty$ . Note that formally we only defined limits for sequences with values in  $\mathbb{R}$  and not with values in  $\mathbb{R} \cup \{-\infty, \infty\}$ . Here we implicitly used the obvious extension, namely we said that the limit of the sequence  $(s_n)$  which

is constantly  $-\infty$  has the limit  $-\infty$ .

Next we consider the case where  $(a_n)$  is divergent to  $+\infty$ . In particular,  $(a_n)$  is not bounded from above and therefore  $+\infty$  is an accumulation value. This is also the only accumulation value, since each subsequence of  $(a_n)$  also diverges to  $+\infty$ . Hence,  $\liminf a_n = +\infty$ . On the other hand for each  $c > 0$  there is an  $N \in \mathbb{N}$  such that  $a_n \geq c$  for all  $n \geq N$ . Therefore  $s_n = \inf\{a_k \mid k \geq n\} \geq c$  for all  $n \geq N$  which shows that also  $(s_n)$  diverges to  $+\infty$ , i.e.  $\lim_{n \rightarrow \infty} s_n = +\infty$ .

Finally we consider the remaining case where  $(a_n)$  is bounded from below and not divergent to  $+\infty$ . Then there exist constants  $c_1, c_2 \in \mathbb{R}$  such that  $c_1 \leq a_n$  for all  $n \in \mathbb{N}$  and  $a_n \leq c_2$  for infinitely many  $n \in \mathbb{N}$ . This implies

$$c_1 \leq s_n = \inf\{a_k \mid k \geq n\} \leq c_2$$

for all  $n \in \mathbb{N}$ , i.e.  $(s_n)$  is bounded. Since  $(s_n)$  is also monotonically increasing as

$$s_{n+1} = \inf\{a_k \mid k \geq n+1\} \geq \min\{\inf\{a_k \mid k \geq n+1\}, a_n\} = \inf\{a_k \mid k \geq n\} = s_n,$$

it must be convergent. Set  $s := \lim_{n \rightarrow \infty} s_n$ . We can recursively define a subsequence  $(a_{n_k})$  of  $(a_n)$  with  $n_1 = 1$  and  $n_{k+1} > n_k$  such that

$$s_{(n_k+1)} = \inf\{a_m \mid m \geq n_k + 1\} \leq a_{n_{k+1}} \leq s_{(n_k+1)} + \frac{1}{k}.$$

Since the right- and left-hand sides of this inequality converge to  $s$  for  $k \rightarrow \infty$ , we also have  $\lim_{k \rightarrow \infty} a_{n_k} = s$  which shows that  $s$  is an accumulation value of  $(a_n)$ . On the other hand, if  $x$  is any other accumulation value of  $(a_n)$  and if  $(a_{j_k})$  is a corresponding subsequence such that  $\lim_{k \rightarrow \infty} a_{j_k} = x$ , then

$$s_{j_k} = \inf\{a_m \mid m \geq j_k\} \leq a_{j_k}$$

shows that  $s = \lim_{k \rightarrow \infty} s_{j_k} \leq \lim_{k \rightarrow \infty} a_{j_k} = x$  which means that  $s$  is indeed the smallest accumulation value of  $(a_n)$ , that is  $\liminf a_n = s$ .

b) Analogous to a).

c) “ $\Rightarrow$ ”: Since the sequence  $(a_n)$  is convergent every subsequence is convergent with the same limit. There exists only one accumulation value and thus  $\liminf a_n = \limsup a_n$ .

“ $\Leftarrow$ ”: Let  $s := \liminf a_n = \limsup a_n$ . Then for all  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $s - \varepsilon < a_n < s + \varepsilon$ . This implies convergence of  $(a_n)_{n \in \mathbb{N}}$  to  $s$ .

d) Let  $s_n := \inf\{a_k : k \geq n\}$ .

“ $\Rightarrow$ ”: We have for any  $c > 0$  an  $N \in \mathbb{N}$  such that  $a_n > c + 1$  for all  $n \geq N$ . Thus  $s_n > c$  for all  $n \geq N$ .

“ $\Leftarrow$ ”: By definition of  $s_n$  we have  $a_n \geq s_n$ . Thus  $a_n \rightarrow \infty$  since  $s_n \rightarrow \infty$ .

e) Analogous to d).

□

#### Exercise 4.

(a)  $(a_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}$ . Then  $\infty$  is the only accumulation value and consequently  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \infty$ .

(b)  $(a_n)_{n \in \mathbb{N}} = ((-1)^n n)_{n \in \mathbb{N}} = (-1, 2, -3, 4, -5, 6, \dots)$ . Then  $\infty$  and  $-\infty$  are the only accumulation values and consequently  $\limsup a_n = \infty$  and  $\liminf a_n = -\infty$ .

(c)  $(a_n)_{n \in \mathbb{N}} = ((-1)^n)_{n \in \mathbb{N}}$ . Then 1 and  $-1$  are the only accumulation values and consequently  $\limsup a_n = 1$  and  $\liminf a_n = -1$ .

(d)  $(a_n)_{n \in \mathbb{N}}$  with

$$a_n = \begin{cases} (-1)^n & : \text{if } n \text{ is divisible by } 3, \\ n & : \text{else.} \end{cases}$$

Then we have  $(a_n)_{n \in \mathbb{N}} = (1, 2, -1, 4, 5, 1, 7, 8, -1, 9, 10, \dots)$  and the set of accumulation values is given by  $\{-1, 1, \infty\}$ . Thus, we have  $\limsup a_n = \infty$  and  $\liminf a_n = -1$ .