Limit Superior and Limit Inferior

Definition 1. Limit superior - limit inferior

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. A number $a \in \mathbb{R} \cup \{\infty, -\infty\}$ is called

- *limit superior of $(a_n)_{n\in\mathbb{N}}$ if a is the largest accumulation value of $(a_n)_{n\in\mathbb{N}}$. In this case, we write*
$$a = \limsup_{n\to\infty} a_n.$$
- *limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the smallest accumulation value of $(a_n)_{n\in\mathbb{N}}$. In this case, we write*
$$a = \liminf_{n\to\infty} a_n.$$

Remark 2.

Almost needless to say, we define the ordering between infinity and real numbers by $-\infty < a < \infty$ for all $a \in \mathbb{R}$. It can be shown that (in contrast to the limit) the limit superior and limit inferior always exist for any real sequence. This will follow from the subsequent results.

Lemma 3.

Let $(a_n)_{n\in\mathbb{R}}$ be a real sequence. Then the following statements hold

a) $\liminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \inf\{a_k \,|\, k \geq n\}$

b) $\limsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sup\{a_k \,|\, k \geq n\}$

c) A sequence is convergent if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \notin \{\pm\infty\}$. In this case holds $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

d) A sequence is divergent to ∞ if and only if $\liminf_{n\to\infty} a_n = \infty$. In this case also holds $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$.

e) A sequence is divergent to $-\infty$ if and only if $\limsup_{n\to\infty} a_n = -\infty$. In this case also holds $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Proof:

a) If (a_n) is not bounded from below, then, $-\infty$ is an accumulation value of (a_n) which necessarily must be the smallest one. So $\liminf a_n = -\infty$. On the other hand, the unboundedness from below of (a_n) implies $s_n := \inf\{a_k \,|\, k \geq n\} = -\infty$ for all $n \in \mathbb{N}$ and therefore also $\lim_{n\to\infty} s_n = -\infty$. Note that formally we only defined limits for sequences with values in $\mathbb{R}$ and not with values in $\mathbb{R} \cup \{-\infty, \infty\}$. Here we implicitely used the obvious extension, namely we said that the limit of the sequence (s_n) which is constantly $-\infty$ has the limit $-\infty$.

Next we consider the case where (a_n) is divergent to $+\infty$. In particular, (a_n) is not bounded from above and therefore $+\infty$ is an accumulation value. This is also the only accumulation value, since each subsequence of (a_n) also diverges to $+\infty$. Hence, $\liminf a_n = +\infty$. On the other hand for each $c > 0$ there is an $N \in \mathbb{N}$ such that $a_n \geq c$ for all $n \geq N$. Therfore $s_n = \inf\{a_k \,|\, k \geq n\} \geq c$ for all $n \geq N$ which shows that also (s_n) diverges to $+\infty$, i.e. $\lim_{n\to\infty} s_n = +\infty$.

Finally we consider the remaining case where (a_n) is bounded from below and not divergent to $+\infty$. Then there exist constants $c_1, c_2 \in \mathbb{R}$ such that $c_1 \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \leq c_2$ for infinitely many $n \in \mathbb{N}$. This implies

$$c_1 \leq s_n = \inf\{a_k \,|\, k \geq n\} \leq c_2$$

for all $n \in \mathbb{N}$, i.e. (s_n) is bounded. Since (s_n) is also monotonically increasing as

$$s_{n+1} = \inf\{a_k \,|\, k \geq n+1\} \geq \min\{\inf\{a_k \,|\, k \geq n+1\}, a_n\} = \inf\{a_k \,|\, k \geq n\} = s_n \ ,$$

it must be convergent. Set $s := \lim_{n\to\infty} s_n$. We can recursively define a subsequence (a_{n_k}) of (a_n) with $n_1 = 1$ and $n_{k+1} > n_k$ such that

$$s_{(n_k+1)} = \inf\{a_m \,|\, m \geq n_k + 1\} \leq a_{n_{k+1}} \leq s_{(n_k+1)} + \frac{1}{k} \ .$$

Since the right- and left-hand sides of this inequality converge to s for $k \to \infty$, we also have $\lim_{k\to\infty} a_{n_k} = s$ which shows that s is an accumulation value of (a_n). On the other hand, if x is any other accumulation value of (a_n) and if (a_{j_k}) is a corresponding subsequence such that $\lim_{k\to\infty} a_{j_k} = x$, then

$$s_{j_k} = \inf\{a_m \mid m \geq j_k\} \leq a_{j_k}$$

shows that $s = \lim_{k\to\infty} s_{j_k} \leq \lim_{k\to\infty} a_{j_k} = x$ which means that s is indeed the smallest accumulation value of (a_n), that is $\liminf a_n = s$.

b) Analogous to a).

c) "$\Rightarrow$": Since the sequence (a_n) is convergent every subsequence is convergent with the same limit. There exists only one accumulation value and thus $\liminf a_n = \limsup a_n$.
"$\Leftarrow$": Let $s := \liminf a_n = \limsup a_n$. Then for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $s - \varepsilon < a_n < s + \varepsilon$. This implies convergence of $(a_n)_{n\in\mathbb{N}}$ to s.

d) Let $s_n := \inf\{a_k : k \geq n\}$.
"$\Rightarrow$": We have for any $c > 0$ an $N \in \mathbb{N}$ such that $a_n > c+1$ for all $n \geq N$. Thus $s_n > c$ for all $n \geq N$.
"$\Leftarrow$": By definition of s_n we have $a_n \geq s_n$. Thus $a_n \to \infty$ since $s_n \to \infty$.

e) Analogous to d).

□

Exercise 4.

(a) $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$. *Then* ∞ *is the only accumulation value and consequently* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$.

(b) $(a_n)_{n\in\mathbb{N}} = ((-1)^n n)_{n\in\mathbb{N}} = (-1, 2, -3, 4, -5, 6, \ldots)$. *Then* ∞ *and* $-\infty$ *are the only accumulation values and consequently* $\limsup a_n = \infty$ *and* $\liminf a_n = -\infty$.

(c) $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$. *Then* 1 *and* -1 *are the only accumulation values and consequently* $\limsup a_n = 1$ *and* $\liminf a_n = -1$.

(d) $(a_n)_{n\in\mathbb{N}}$ *with*

$$a_n = \begin{cases} (-1)^n & : \textit{if } n \textit{ is divisible by } 3, \\ n & : \textit{else.} \end{cases}$$

Then we have $(a_n)_{n\in\mathbb{N}} = (1, 2, -1, 4, 5, 1, 7, 8, -1, 9, 10, \ldots)$ *and the set of accumulation values is given by* $\{-1, 1, \infty\}$. *Thus, we have* $\limsup a_n = \infty$ *and* $\liminf a_n = -1$.