Exercise 1. Non-Riemann integrable function

Consider the function

$$f : [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is not Riemann integrable.

Exercise 2. Integrability of monotonic functions

Let $f : [a,b] \to \mathbb{R}$ be monotonically increasing. That means that $x, y \in [a,b]$, $x \leq y$ implies $f(x) \leq f(y)$. Show that $f \in \mathcal{R}([a,b])$.

Exercise 3. Riemann integrable function

Let's define the function

$$f : [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q} \text{ and } \gcd(p,q) = 1, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable and that

$$\int_0^1 f(x)\, dx = 0\,.$$

Hint: *Choose suitable partitions to calculate the upper sum and lower sum.*