**The Bright Side of Mathematics**

The following pages cover the whole Real Analysis course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

The Bright Side of Mathematics

# Real Analysis – Part 1

Calculus, analysis, infinitesimal calculus, ...

**Goal:** Understanding differential and integral calculations $\frac{df}{dx}$, $\int_a^b f\,dx$

**Topics:** Limits, continuity, derivatives, integrals.

**Foundation:** Real numbers! $\mathbb{R}$ $\rightsquigarrow$ Start Learning Mathematics

↳ Start Learning Reals

**Axioms of the reals:** A non-empty set $\mathbb{R}$ together with operations $+$, $\cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x\cdot(y+z) = x\cdot y + x\cdot z$

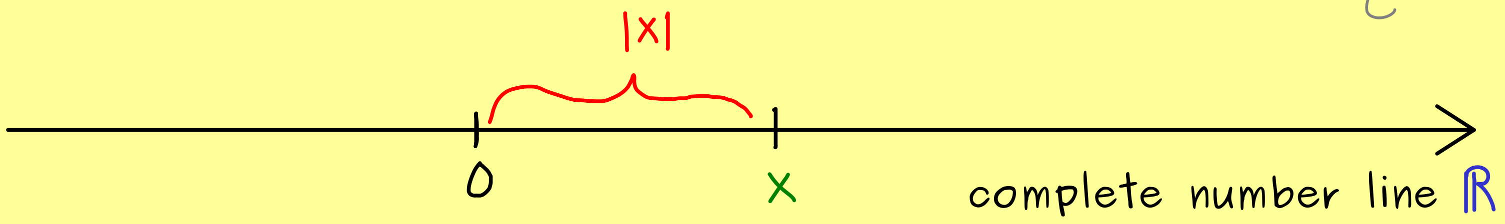
(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Absolute value:

$|x\cdot y| = |x|\cdot|y|$

$|x+y| \leq |x| + |y|$



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## Real Analysis – Part 2

**Sequences:** A sequence of real numbers:

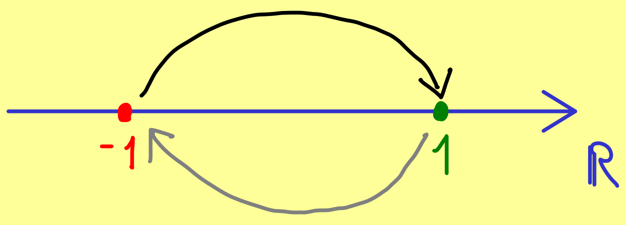
a map $a: \mathbb{N} \longrightarrow \mathbb{R}$

or $a: \mathbb{N}_0 \longrightarrow \mathbb{R}$

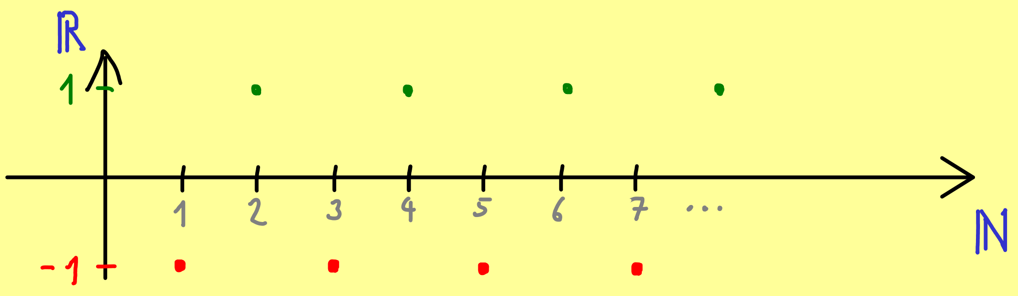
Notations: $(a_1, a_2, a_3, \dots)$ infinite list of numbers

$(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n=1}^{\infty}$ or $(a_n)$

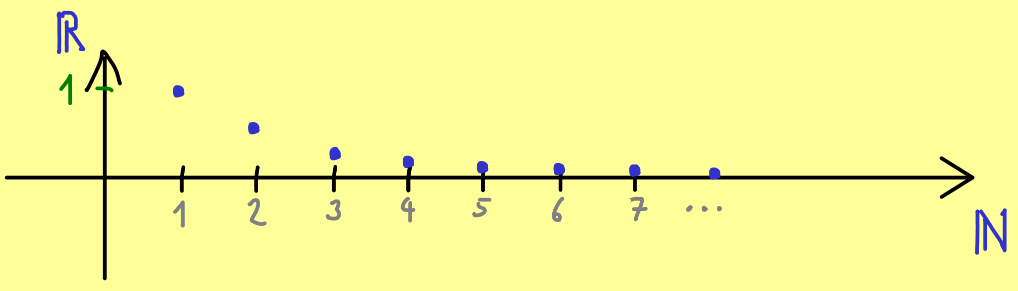
**Examples:** (a) $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\right)_{n\in\mathbb{N}} = (-1, 1, -1, 1, \dots)$




Graph:




(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots\right)$

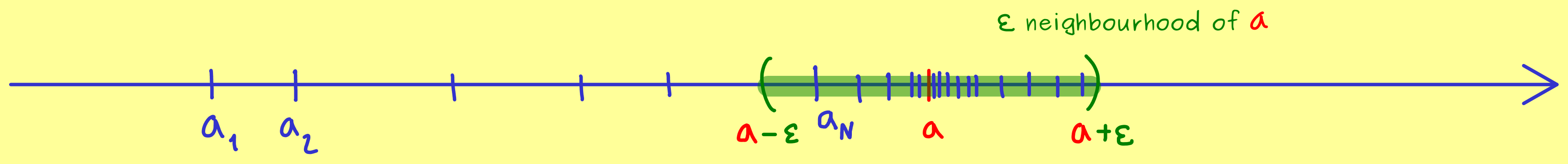



We will see: $\lim\limits_{n\to\infty} a_n = 0$

(c) $(a_n)_{n\in\mathbb{N}} = (2^n)_{n\in\mathbb{N}} = (2, 4, 8, 16, 32, 64, 128, 256, \dots)$

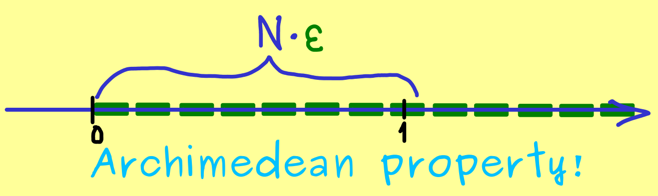
**Definition:** A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad |a_n - a| < \varepsilon$$




If there is no such $a \in \mathbb{R}$, we call the sequence $(a_n)_{n\in\mathbb{N}}$ divergent.

**Example:** $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is convergent to $0 \in \mathbb{R}$.




**Proof:** Let $\varepsilon > 0$. We choose $N \in \mathbb{N}$ such that $N \cdot \varepsilon > 1$.

Then for $n \geq N$, we have: $|a_n - 0| = |a_n| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

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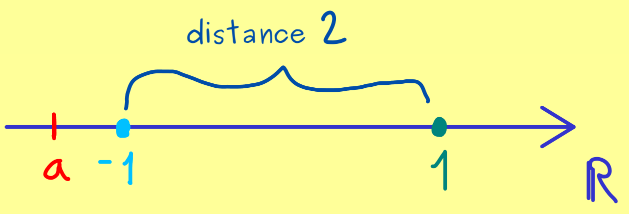
## Real Analysis – Part 3

**Example:** $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\right)_{n\in\mathbb{N}}$ is divergent.

**Proof:** Assume the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent to $a\in\mathbb{R}$.

$$\forall \varepsilon>0 \;\; \exists N\in\mathbb{N} \;\; \forall n\geq N : \quad |a_n - a| < \varepsilon$$




Choose: $\varepsilon = 1$ Then: $|a_N - a| < \varepsilon$

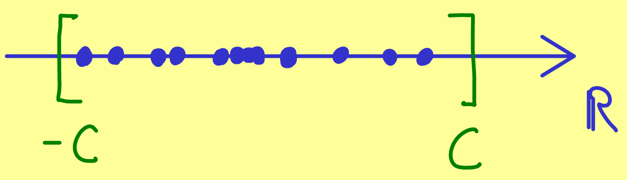
and $|a_{N+1} - a| < \varepsilon$

Hence: $|1 - a| < \varepsilon$ and $|(-1) - a| < \varepsilon$

$$\underline{2} = |1-(-1)| = |1 - a + a - (-1)| \leq |1-a| + |a-(-1)| = |1-a| + |(-1)-a| \underline{< 2} \quad ↯$$

**Definition:** A sequence $(a_n)_{n\in\mathbb{N}}$ is called bounded if

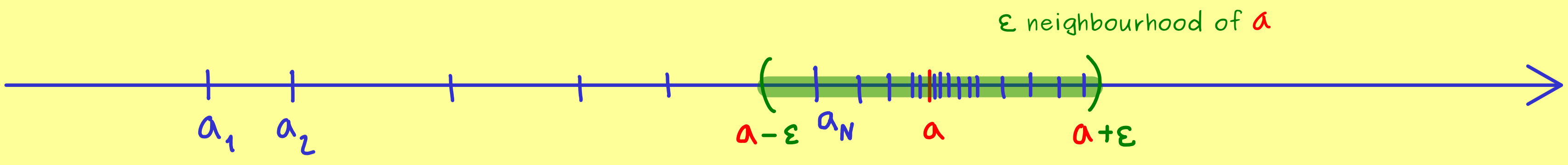
$$\exists C\in\mathbb{R} \;\; \forall n\in\mathbb{N} : \quad |a_n| \leq C$$




Otherwise, the sequence is called unbounded.

**Important fact:** $(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ bounded

**Proof:** There is $a\in\mathbb{R}$ with:

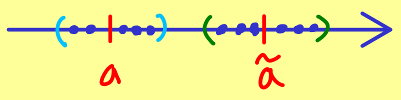



$$C := \max\left(|a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|, |a|+\varepsilon\right)$$

**Important fact:** $(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ There is only one limit $a\in\mathbb{R}$

$$a = \lim_{n\to\infty} a_n$$

**Proof:** Assume there are two limits $a \neq \tilde{a}$. $\varepsilon := \frac{1}{4}|a-\tilde{a}| > 0$




Then: $\exists N\in\mathbb{N} \;\; \forall n\geq N : \quad |a_n - a| < \varepsilon$

$\exists \tilde{N}\in\mathbb{N} \;\; \forall n\geq \tilde{N} : \quad |a_n - \tilde{a}| < \varepsilon$

Therefore: For $n \geq \max(N,\tilde{N})$: $|a-\tilde{a}| = |a - a_n + a_n - \tilde{a}| \leq \overbrace{|a-a_n|}^{<\varepsilon} + \overbrace{|a_n-\tilde{a}|}^{<\varepsilon}$

$$< \frac{1}{2}|a-\tilde{a}| \quad ↯$$

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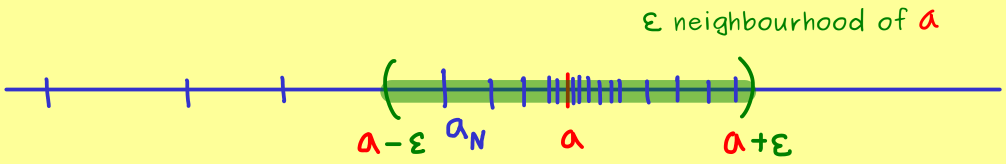
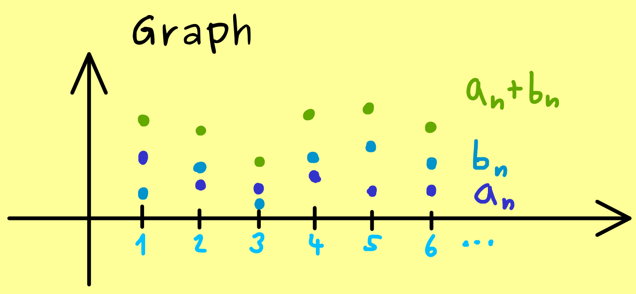
## Real Analysis – Part 4

$(a_n)_{n\in\mathbb{N}}$ convergent to $a \in \mathbb{R}$: $\lim\limits_{n\to\infty} a_n = a$




$a_n \xrightarrow{n\to\infty} a$




**Theorem on limits:** $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequences.

Then: (a)

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

(b)

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(c)

$$\lim_{n\to\infty}\left(\frac{a_n}{\underbrace{b_n}_{\neq 0}}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n \neq 0}$$

**Example:** $c_n = \dfrac{2n^2 + 5n - 1}{-5n^2 + n + 1}$ convergent? limit?

We know: $\frac{1}{n} \xrightarrow{n\to\infty} 0$

By (b): $\frac{1}{n}\cdot\frac{1}{n} \xrightarrow{n\to\infty} 0$

$$= \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{2n^2 + 5n - 1}{-5n^2 + n + 1} = \frac{2 + \frac{5}{n} - \frac{1}{n^2}}{-5 + \frac{1}{n} + \frac{1}{n^2}} \xrightarrow[\text{with limit theorems}]{n\to\infty} \frac{2+0-0}{-5+0+0} = -\frac{2}{5}$$

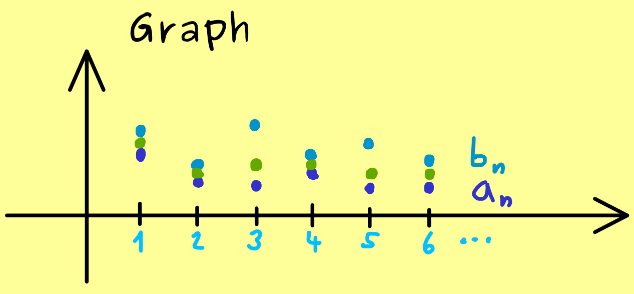
# Real Analysis – Part 5

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequences.

$$\Longrightarrow \lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \,, \quad \lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

In particular: $\lim_{n\to\infty}(a \cdot b_n) = a \cdot \lim_{n\to\infty}(b_n)$

Graph




<u>Properties:</u> $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequences.

(a) Monotonicity: $a_n \leq b_n$ for all $n \in \mathbb{N}$

$$\Longrightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem: $a_n \leq c_n \leq b_n$ for all $n\in\mathbb{N}$ <u>and</u> $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$$\Longrightarrow (c_n)_{n\in\mathbb{N}} \text{ convergent with } \lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

<u>Proof of (b):</u> $(b_n - a_n) \xrightarrow{n\to\infty} \underbrace{\lim_{n\to\infty} b_n}_{b} - \underbrace{\lim_{n\to\infty} a_n}_{a} = 0$ (by the limit theorems)

$d_n := c_n - a_n \quad \Longrightarrow \quad 0 \leq d_n \leq b_n - a_n$

Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ with: $\forall n \geq N: \; |b_n - a_n| < \varepsilon$

$|d_n - 0| \leq$

$\Longrightarrow (d_n)_{n\in\mathbb{N}}$ is convergent with limit $0$

limit theorems

$\Longrightarrow (c_n)_{n\in\mathbb{N}} = (d_n + a_n)_{n\in\mathbb{N}}$ is convergent with limit $a$ $\square$

<u>Example:</u> sequence $(c_n)_{n\in\mathbb{N}}$ given by $c_n = \sqrt{n^2+1} - n$ convergent? limit?

$$= \left(\sqrt{n^2+1} - n\right) \cdot \frac{\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n}$$

$$= \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\underbrace{\sqrt{n^2+1}}_{>0} + n} \leq \frac{1}{n}$$

$$\Longrightarrow 0 \leq c_n \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \underset{\text{Sandwich}}{\Longrightarrow} \lim_{n\to\infty} c_n = 0$$

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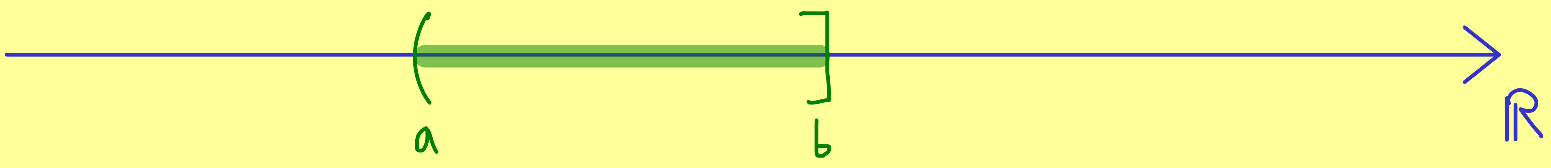
# Real Analysis – Part 6

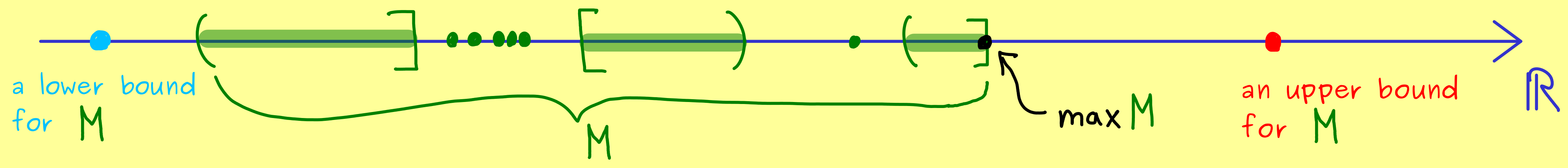



interval: $(a,b] := \{ x \in \mathbb{R} \mid a < x \leq b \}$

$[a,b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \}$

$[a,\infty) := \{ x \in \mathbb{R} \mid a \leq x \}$

$(-\infty, b) := \{ x \in \mathbb{R} \mid x < b \}$



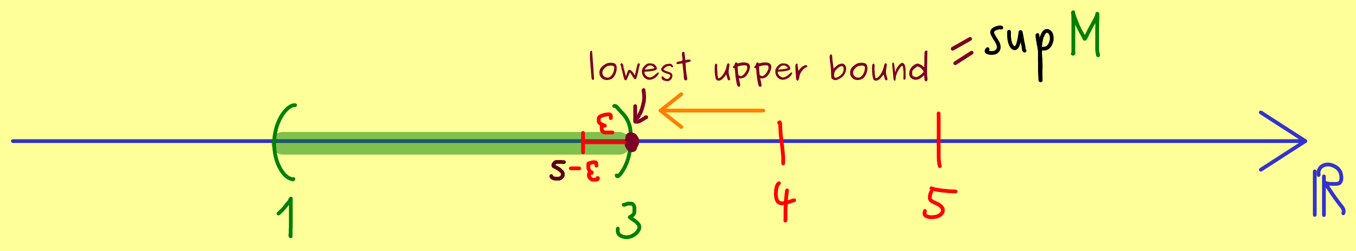

**Definition:** For a subset $M \subseteq \mathbb{R}$: $b \in \mathbb{R}$ is called <u>an upper bound for M</u> if $\forall x \in M: \; x \leq b$

$a \in \mathbb{R}$ is called <u>a lower bound for M</u> if $\forall x \in M: \; x \geq a$

If $b$ is an upper bound for $M$ <u>and</u> $b \in M$, then $b$ is called a <u>maximal element of M</u>.

If $a$ is a lower bound for $M$ <u>and</u> $a \in M$, then $a$ is called a <u>minimal element of M</u>.
min M

**Example:**
- $M = [1,3]$, $\max M = 3$ $\quad \min M = 1$
- $M = (1,3)$, $\max M$, $\min M$ do not exist $\rightsquigarrow$ $\sup M$, $\inf M$




**Definition:** For a subset $M \subseteq \mathbb{R}$: $s \in \mathbb{R}$ is called <u>supremum of M</u> if
- $\forall x \in M: \; x \leq s$ (upper bound for $M$)
- $\forall \varepsilon > 0 \; \exists \tilde{x} \in M: \; s - \varepsilon < \tilde{x}$ ($s - \varepsilon$ is no upper bound for $M$)

Then write: $\sup M := s$ or $\sup M := \infty$ if $M$ is not bounded from above

or $\sup \emptyset := -\infty$

For a subset $M \subseteq \mathbb{R}$: $\ell \in \mathbb{R}$ is called <u>infimum of M</u> if
- $\forall x \in M: \; x \geq \ell$ (lower bound for $M$)
- $\forall \varepsilon > 0 \; \exists \tilde{x} \in M: \; \ell + \varepsilon > \tilde{x}$ ($\ell + \varepsilon$ is no lower bound for $M$)

Then write: $\inf M := \ell$ or $\inf M := -\infty$ if $M$ is not bounded from below

or $\inf \emptyset := \infty$

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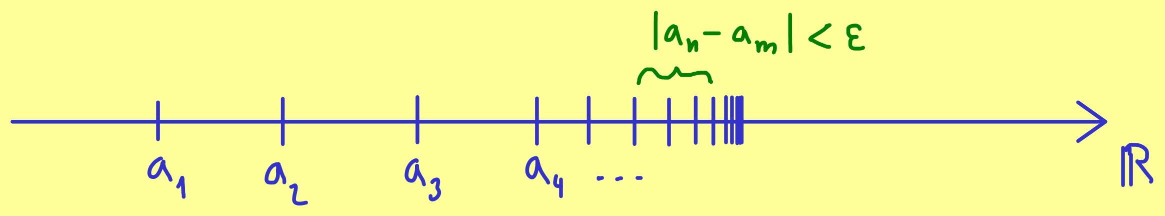
# Real Analysis – Part 7

$(a_n)_{n\in\mathbb{N}}$ convergent (there is a limit $a = \lim_{n\to\infty} a_n$ )

Different idea:




Definition: If $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n, m \geq N : \; |a_n - a_m| < \varepsilon$ ,

then $(a_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence.

Important fact: For a sequence of real numbers:

Completeness axiom (C)

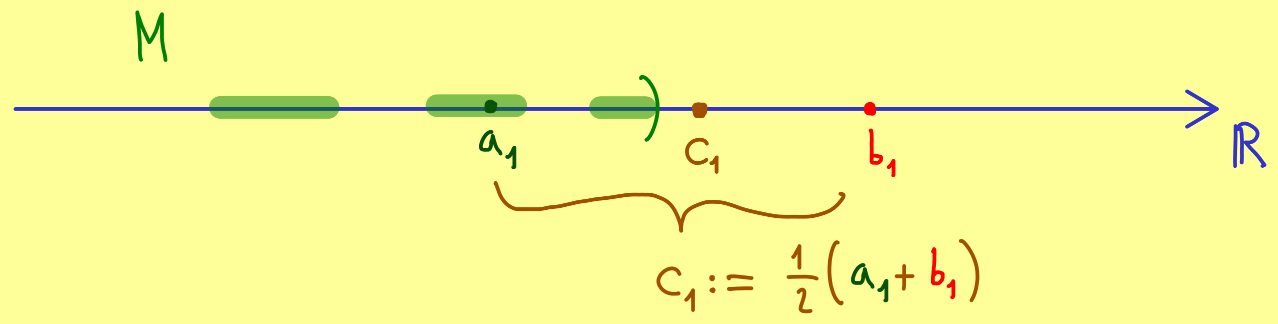
Cauchy sequence $\Longleftrightarrow$ Convergent sequence

Start Learning Real – Part 2

Dedekind completeness: If $M \subseteq \mathbb{R}$ is bounded from above, then $\sup M \in \mathbb{R}$ (exists)

If $M \subseteq \mathbb{R}$ is bounded from below, then $\inf M \in \mathbb{R}$ (exists)

Proof:




Two cases: (1) $c_1$ is an upper bound for $M$ : $b_2 := c_1$ , $a_2 := a_1$

(2) $c_1$ is not an upper bound for $M$ : $\exists x \in M : \; x > c_1$

$a_2 := x$ , $b_2 := b_1$

$c_n := \frac{1}{2}(a_n + b_n)$

Two cases: (1) $c_n$ is an upper bound for $M$ : $b_{n+1} := c_n$ , $a_{n+1} := a_n$

(2) $c_n$ is not an upper bound for $M$ : $\exists x \in M : \; x > c_n$

$a_{n+1} := x$ , $b_{n+1} := b_n$

For $m > n$ : $|b_n - b_m| \leq |b_n - a_n| \leq \underbrace{\left(\frac{1}{2}\right)^{n-1} |b_1 - a_1|}_{\text{gets arbitrarily small}}$

$\Longrightarrow (b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence

$\Longrightarrow (b_n)_{n\in\mathbb{N}}$ is a convergent sequence with limit $\sup M$

Important application: If $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing $(a_{n+1} \leq a_n$ for all $n)$

and bounded from below (the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound),

then: $(a_n)_{n\in\mathbb{N}}$ is convergent.

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# Real Analysis – Part 8

**Fact:** If $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing $(a_{n+1} \geq a_n$ for all $n)$ and bounded from above (the set $\{a_n\}_{n\in\mathbb{N}}$ has an upper bound), then: $(a_n)_{n\in\mathbb{N}}$ is convergent.

(Monotone convergence criterion)

**Example:** The sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

**Proof:** (1) Monotonicity: $\frac{a_{n+1}}{a_n}$ $\begin{pmatrix} \leq 1 & \text{mon. decreasing} \\ \geq 1 & \text{mon. increasing} \end{pmatrix}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \left(1+\frac{1}{n}\right)\cdot\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} = \left(1+\frac{1}{n}\right)\left(\frac{\left(1+\frac{1}{n+1}\right)^{n(n+1)}}{\left(1+\frac{1}{n}\right)^{n(n+1)}}\right)$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n(n+1)+n+1-1}{n(n+1)+n+1}\right)^{n+1} = \left(1+\frac{1}{n}\right)\left(1 - \frac{1}{\underbrace{n^2+2n+1}_{(n+1)^2}}\right)^{n+1}$$

Bernoulli's inequality:
For $k\in\mathbb{N}$ and $x \geq -1$
$$(1+x)^k \geq 1 + k\cdot x$$

$$\geq \left(1+\frac{1}{n}\right)\left(1 + \cancel{(n+1)}\cdot\left(-\frac{1}{(n+1)^{\cancel{2}}}\right)\right)$$

$$= \left(\frac{\cancel{n+1}}{\cancel{n}}\right)\cdot\left(\frac{\cancel{n}}{\cancel{n+1}}\right) = 1 \quad \checkmark$$

(2) Bounded from above: $a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\frac{1}{n}\right)^k$

$$= \underbrace{\binom{n}{0}}_{=1}\cdot 1^n\cdot\underbrace{\left(\frac{1}{n}\right)^0}_{=1} + \underbrace{\binom{n}{1}}_{n}\cdot 1^{n-1}\left(\frac{1}{n}\right)^1 + \sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= 1 + 1 + \sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \leq 2 + 1 - \frac{1}{n} \leq 3$$

We have: $\binom{n}{k}\cdot\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)!\cdot k!}\cdot\left(\frac{1}{n^k}\right) = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{n\cdot \underbrace{n}_{\leq 1}\cdot n \cdots n}\cdot\frac{1}{k!} \leq \frac{1}{k!}$

$$\leq \frac{1}{k\cdot(k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \text{and} \quad \sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right) \overset{\text{telescoping}}{=} 1 - \frac{1}{n}$$

$\xRightarrow[\text{Monotone convergence criterion}]{\text{fact}}$ The sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =: e \qquad \text{Euler's number}$$

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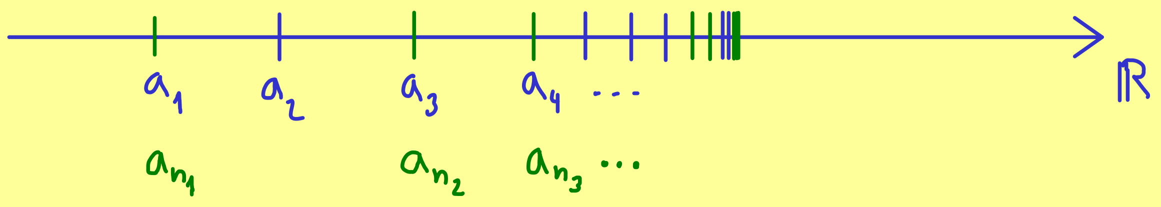
## Real Analysis – Part 9




**Definition:** Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers that is strictly monotonically increasing, then $(a_{n_k})_{k\in\mathbb{N}}$ is called a subsequence of $(a_n)_{n\in\mathbb{N}}$. $\quad (\forall k\in\mathbb{N}: \; n_{k+1} > n_k)$

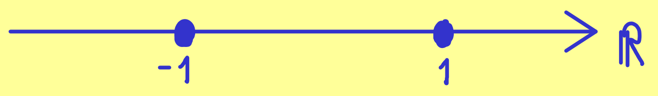
**Example:** $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$, $\quad$ $n = 1, 2, 3, 4, 5, 6, 7, 8, \dots$ $\quad \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \dots\right)$

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2^k})_{k\in\mathbb{N}} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots\right)$$

**Fact:** $(a_n)_{n\in\mathbb{N}}$ convergent with $\lim\limits_{n\to\infty} a_n = a$

$\Longrightarrow$ every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent $\lim\limits_{k\to\infty} a_{n_k} = a$

**Example:** $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$



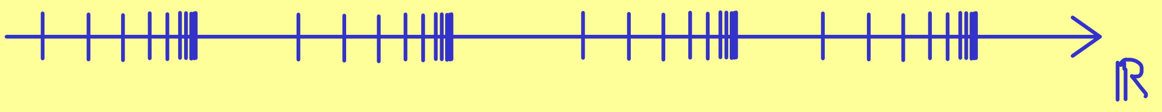

subsequence: $(a_{n_k})_{k\in\mathbb{N}} = (a_{2\cdot k})_{k\in\mathbb{N}} = (1, 1, 1, 1, 1, \dots)$ $\quad$ limit $1$

accumulation values

subsequence: $(a_{n_k})_{k\in\mathbb{N}} = (a_{2\cdot k+1})_{k\in\mathbb{N}} = (-1, -1, -1, \dots)$ $\quad$ limit $-1$

**Definition:** $a\in\mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$

(cluster point, accumulation point, limit point, partial limit, ...)




**Show:** $a\in\mathbb{R}$ is an accumulation value of $(a_n)_{n\in\mathbb{N}}$

$\Longleftrightarrow$ $\forall \varepsilon > 0$ : The $\varepsilon$-neighbourhood of $a$ contains infinitely many sequence members of $(a_n)_{n\in\mathbb{N}}$ $\quad (= (a-\varepsilon, a+\varepsilon))$

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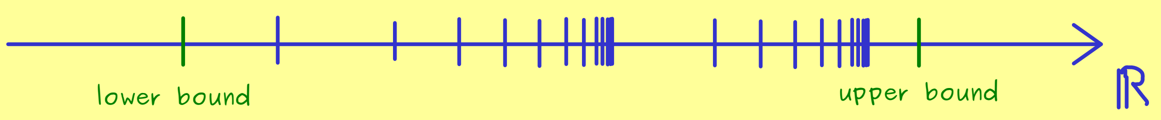
# Real Analysis – Part 10

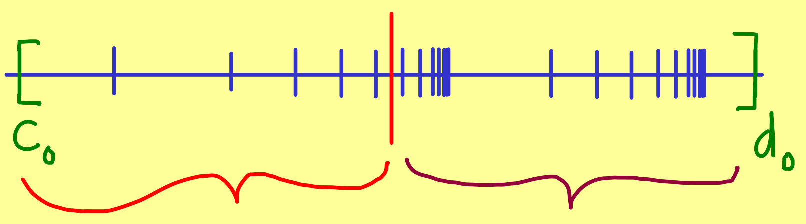
## Bolzano–Weierstrass theorem

$(a_n)_{n\in\mathbb{N}}$ bounded $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value

(has a convergent subsequence)




**Proof:**

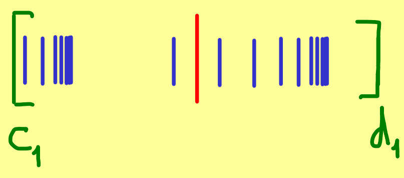



If infinitely many sequence members in it: Choose left–hand interval

Otherwise: Choose right–hand interval

repeat

New interval:




We get: $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2] \supset [c_3, d_3] \supset \dots$

And: $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$ , $d_2 - c_2 = \frac{1}{2}(d_1 - c_1) = \frac{1}{4}(d_0 - c_0)$, ...

$d_n - c_n = \frac{1}{2^n}(d_0 - c_0) \xrightarrow{n\to\infty} 0$

We know: $(c_n)_{n\in\mathbb{N}}$ mon. increasing and bounded, $(d_n)_{n\in\mathbb{N}}$ mon. decreasing and bounded $\Big\}\Longrightarrow$ $(c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ convergent

By limit theorems: $0 = \lim\limits_{n\to\infty}(d_n - c_n) = \lim\limits_{n\to\infty} d_n - \lim\limits_{n\to\infty} c_n$

Define a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ by choosing $a_{n_k} \in [c_k, d_k]$

$\Longrightarrow c_k \le a_{n_k} \le d_k$

$\overset{\text{Sandwich theorem}}{\Longrightarrow}$ $(a_{n_k})_{k\in\mathbb{N}}$ is convergent

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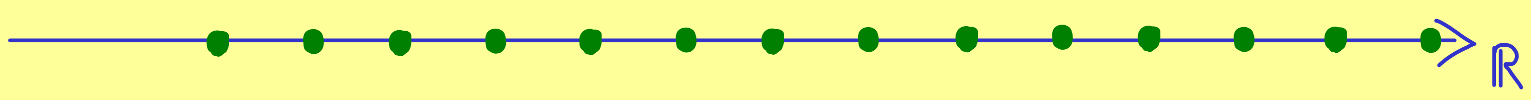
# Real Analysis – Part 11

Example: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = n$




$\lim\limits_{n\to\infty} a_n = \infty$ $:\Longleftrightarrow$ divergent to $\infty$ $:\Longleftrightarrow$ $\forall C>0 \;\; \exists N\in\mathbb{N} \;\; \forall n\geq N : a_n > C$

$\lim\limits_{n\to\infty} a_n = -\infty$ $:\Longleftrightarrow$ divergent to $-\infty$ $:\Longleftrightarrow$ $\forall C<0 \;\; \exists N\in\mathbb{N} \;\; \forall n\geq N : a_n < C$

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $\infty$ $:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from above

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $-\infty$ $:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from below

A given sequence $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values:

and none, one or two improper accumulation values

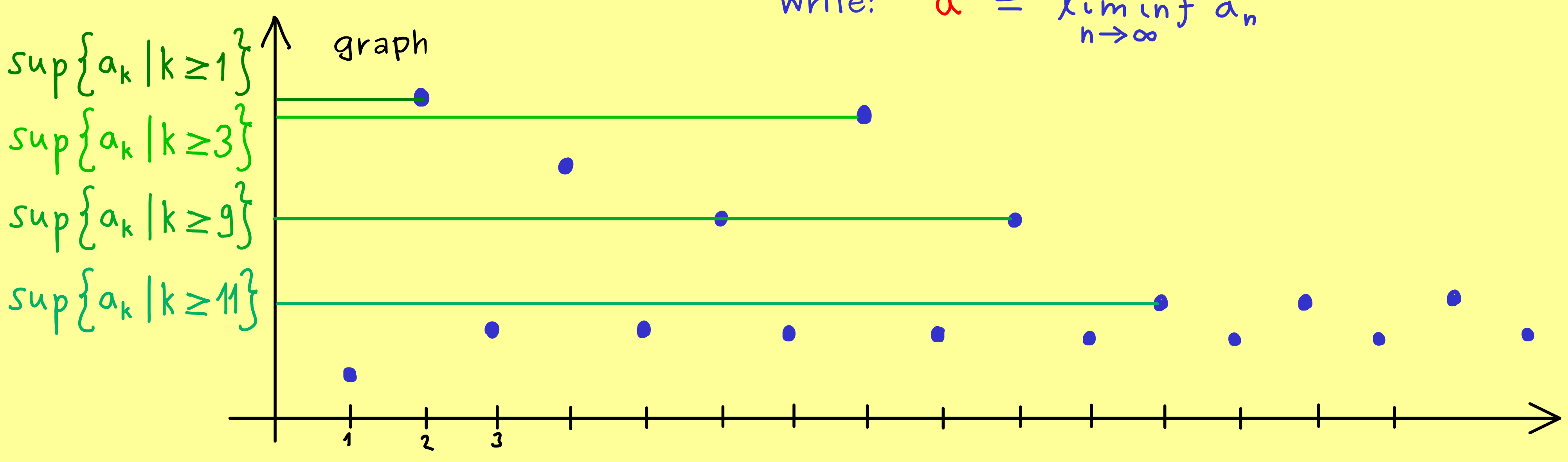

Definition: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. An element $a\in\mathbb{R}\cup\{-\infty,\infty\}$ is called:

- limit superior of $(a_n)_{n\in\mathbb{N}}$ if $a$ is the largest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

  Write: $a = \limsup\limits_{n\to\infty} a_n$

- limit inferior of $(a_n)_{n\in\mathbb{N}}$ if $a$ is the smallest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

  Write: $a = \liminf\limits_{n\to\infty} a_n$




Fact: $\limsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sup\{a_k \mid k\geq n\}$

$\liminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \inf\{a_k \mid k\geq n\}$

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## Real Analysis – Part 12

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

$$\Longrightarrow \limsup_{n\to\infty} a_n \,,\, \liminf_{n\to\infty} a_n \in \mathbb{R}\cup\{\pm\infty\} = [-\infty,\infty]$$

Example: $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\cdot n\right)_{n\in\mathbb{N}} = (-1,2,-3,4,-5,\dots)$

$$\limsup_{n\to\infty} a_n = \infty$$
$$\liminf_{n\to\infty} a_n = -\infty$$

Properties: (a) $(a_n)_{n\in\mathbb{N}}$ is convergent $\Longleftrightarrow$ $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \notin \{\pm\infty\}$

(b) $(a_n)_{n\in\mathbb{N}}$ is divergent to $\infty$ $\Longleftrightarrow$ $\left(\limsup_{n\to\infty} a_n =\right) \liminf_{n\to\infty} a_n = \infty$

(c) $(a_n)_{n\in\mathbb{N}}$ is divergent to $-\infty$ $\Longleftrightarrow$ $\left(\liminf_{n\to\infty} a_n =\right) \limsup_{n\to\infty} a_n = -\infty$

(d) For $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, we have:

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

If $a_n, b_n \geq 0$: $$\limsup_{n\to\infty}(a_n\cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n$$

(only if the right-hand side is defined)
$\infty - \infty$ not defined
$0\cdot\infty$ not defined

$$\liminf_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

If $a_n, b_n \geq 0$: $$\liminf_{n\to\infty}(a_n\cdot b_n) \geq \liminf_{n\to\infty} a_n \cdot \liminf_{n\to\infty} b_n$$

Example: $(a_n)_{n\in\mathbb{N}} = (1,-1,1,-1,1,-1,1,-1,\dots)$

$(b_n)_{n\in\mathbb{N}} = (0,2,0,2,0,2,0,2,\dots)$

$(a_n+b_n)_{n\in\mathbb{N}} = (1,1,1,1,1,1,1,1,\dots)$

$$1 = \limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 1+2 = 3$$
$$1 = \liminf_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n = -1+0 = -1$$

Example: $(a_n)_{n\in\mathbb{N}} = (1,0,1,0,1,0,1,0,\dots)$

$(b_n)_{n\in\mathbb{N}} = (0,2,0,2,0,2,0,2,\dots)$

$(a_n\cdot b_n)_{n\in\mathbb{N}} = (0,0,0,0,0,0,0,0,\dots)$

$$0 = \limsup_{n\to\infty}(a_n\cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n = 1\cdot 2 = 2$$

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## Real Analysis – Part 13

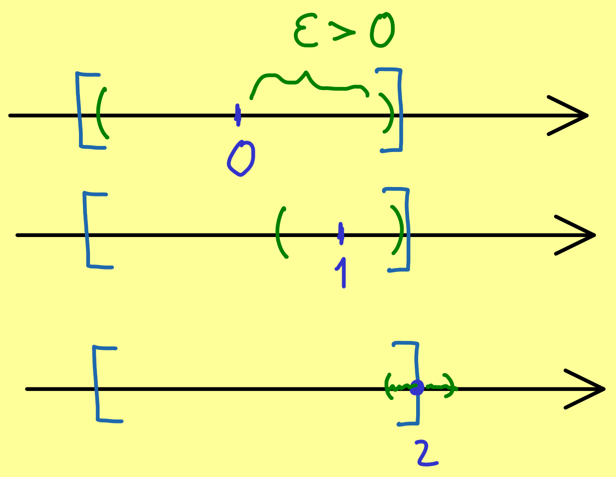

For $\varepsilon > 0$: $\quad (x-\varepsilon, x+\varepsilon) =: B_\varepsilon(x)$ $\varepsilon$-neighbourhood of $x$

$M \subseteq \mathbb{R}$ is called a neighbourhood of $x$ if there is $\varepsilon > 0$ such that $M \supseteq B_\varepsilon(x)$

Example: $[-2,2]$ is a neighbourhood of $0$

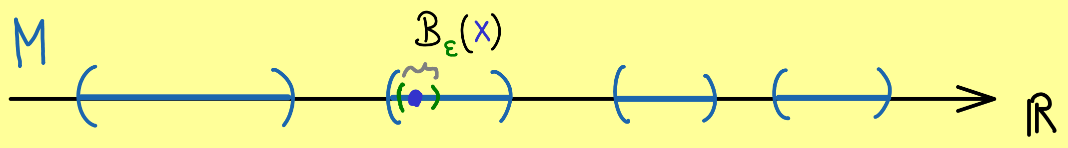
$[-2,2]$ is a neighbourhood of $1$

$[-2,2]$ is not a neighbourhood of $2$




Definition: $M \subseteq \mathbb{R}$ is called open (in $\mathbb{R}$) if, for all $x \in M$, $M$ is a neighbourhood of $x$.

$$\forall x \in M \quad \exists \varepsilon > 0 : \quad M \supseteq B_\varepsilon(x)$$

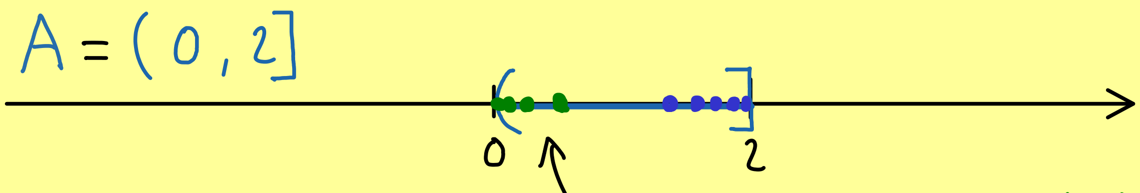



$A \subseteq \mathbb{R}$ is called closed (in $\mathbb{R}$) if $A^c := \mathbb{R} \setminus A$ is open.

Example:
- $\emptyset$, $\mathbb{R}$ are both open and closed.
- $[-2,2]$ is closed but not open.
- $(-2,2)$ is open but not closed.
- $(-2,2]$ is neither open nor closed.

Fact: $A \subseteq \mathbb{R}$ is closed $\iff$ For all convergent sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n \in \mathbb{N}$, we have: $\lim\limits_{n\to\infty} a_n \in A$

Example: $A = (0,2]$




Take $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

Then: $\lim\limits_{n\to\infty} a_n = 0 \notin (0,2]$

Definition: $A \subseteq \mathbb{R}$ is called compact if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n \in \mathbb{N}$, there is a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} \in A$.

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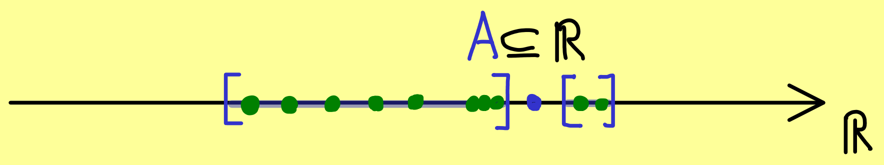
# Real Analysis – Part 14

Compact set (sequentially compact set):




Any sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ has an accumulation value $a \in A$.

**Example:** (a) $\emptyset$ is compact.

(b) $\{5\}$ is compact.

(c) $\mathbb{R}$ is not compact. $(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ has no accumulation value $a\in\mathbb{R}$

(d) $[c,d]$, $c \leq d$, compact set.

Let $(a_n)_{n\in\mathbb{N}} \subseteq [c,d] \Rightarrow (a_n)_{n\in\mathbb{N}}$ is bounded

$\overset{\text{Bolzano-Weierstrass theorem}}{\Longrightarrow}$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value $a \in \mathbb{R}$

$\overset{[c,d] \text{ closed}}{\Longrightarrow}$ accumulation value actually satisfies $a \in [c,d]$

**Heine-Borel theorem** For $A \subseteq \mathbb{R}$, we have:

$A$ is compact $\Longleftrightarrow$ $A$ is bounded and closed

**Proof:** $(\Leftarrow)$ Do the same as before with Bolzano-Weierstrass theorem.

$(\Rightarrow)$ Assume $A$ is compact.

Let $(a_n)_{n\in\mathbb{N}} \subseteq A$ be a convergent sequence with limit $\tilde{a} \in \mathbb{R}$.

$\overset{A \text{ is compact}}{\Longrightarrow}$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value $a \in A$.

$\overset{\text{only one acc. value}}{\Longrightarrow}$ $\tilde{a} = a \in A$ $\Longrightarrow$ $A$ is closed.

Assume $A$ is not bounded.

$\Longrightarrow$ There is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ with $|a_n| > n$ for all $n \in \mathbb{N}$.

$\Longrightarrow$ no accumulation value $\Longrightarrow$ $A$ is not compact.

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## Real Analysis – Part 15

Series: "infinite sum", special sequence

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{k=1}^{\infty} a_k$$

Example: sequence $(a_k)_{k\in\mathbb{N}} = \left((-1)^k\right)_{k\in\mathbb{N}}$

$$\sum_{k=1}^{\infty} a_k = (\underbrace{-1 + 1}_{=0}) + (\underbrace{(-1) + 1}_{=0}) + (\underbrace{(-1) + 1}_{=0}) + (\underbrace{(-1) + 1}_{=0}) + (-1) + \cdots \overset{?}{=} 0$$

$$\sum_{k=1}^{\infty} a_k = -1 + (\underbrace{1 + (-1)}_{=0}) + (\underbrace{1 + (-1)}_{=0}) + (\underbrace{1 + (-1)}_{=0}) + (\underbrace{1 + (-1)}_{=0}) + \cdots \overset{?}{=} -1$$

Definition: Let $(a_k)_{k\in\mathbb{N}}$ be a sequence. The sequence $(S_n)_{n\in\mathbb{N}}$ given by

$$S_n := \sum_{k=1}^{n} a_k$$

is called a series.

If $(S_n)_{n\in\mathbb{N}}$ is convergent, we write:

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Example from above: $\left(\sum_{k=1}^{n} (-1)^k\right)_{n\in\mathbb{N}} = (-1, 0, -1, 0, -1, 0, -1, \ldots)$

not convergent!

Another example: $\left(\sum_{k=1}^{n} (1)^k\right)_{n\in\mathbb{N}} = (1, 2, 3, 4, \ldots)$ divergent to $\infty$

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## Real Analysis – Part 16

**Series:** $\sum_{k=1}^{\infty} a_k$ is the sequence of partial sums $\sum_{k=1}^{n} a_k$

**Example:** geometric series $\sum_{k=0}^{\infty} q^k$ , $q \in \mathbb{R}$

We show: $\sum_{k=0}^{\infty} q^k$ convergent $\Longleftrightarrow$ $|q| < 1$

Question: $s_n = \sum_{k=0}^{n} q^k = ?$

For $q \neq 1$: $(1-q) \cdot \sum_{k=0}^{n} q^k = \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1} = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$

$$s_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

$(s_n)_{n \in \mathbb{N}}$ convergent $\Longleftrightarrow$ $(q^n)_{n \in \mathbb{N}}$ convergent to $0$ $\Longleftrightarrow$ $|q| < 1$

For $|q| < 1$: $\boxed{\sum_{k=0}^{\infty} q^k = \lim_{n \to \infty} s_n = \frac{1}{1-q}}$ geometric series

**Example:** Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty \quad \text{(divergent to infinity)}$$

**Proof:** $s_n = \sum_{k=1}^{n} \frac{1}{k}$ (sequence is monotonically increasing)

Show that $(s_n)_{n \in \mathbb{N}}$ is not bounded from above.

$$s_{2^m} = s_1 + (s_2 - s_1) + (s_4 - s_2) + (s_8 - s_4) + \cdots + (s_{2^m} - s_{2^{m-1}})$$

$$= s_1 + \sum_{j=1}^{m} (s_{2^j} - s_{2^{j-1}}) > s_1 + m \cdot \frac{1}{2} \xrightarrow{m \to \infty} \infty$$

because:

$$s_{2^j} - s_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j} = 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$$

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# Real Analysis – Part 17

**Series:** $\sum_{k=1}^{\infty} a_k$ sequence of partial sums

**Properties:** If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, $\lambda \in \mathbb{R}$, then:

(a) $\sum_{k=1}^{\infty} (a_k + b_k)$ is also convergent

and the limit is: $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

(b) $\sum_{k=1}^{\infty} (\lambda \cdot a_k)$ is also convergent

and the limit is: $\sum_{k=1}^{\infty} (\lambda \cdot a_k) = \lambda \cdot \sum_{k=1}^{\infty} a_k$

**Cauchy criterion:** $\sum_{k=1}^{\infty} a_k$ is convergent $\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq m \geq N :$

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

**Proof:** $S_n := \sum_{k=1}^{n} a_k$ . $(S_n)_{n \in \mathbb{N}}$ is convergent $\overset{\text{completeness}}{\Longleftrightarrow}$ $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence

$\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall \tilde{n}, \tilde{m} \geq N : \ |S_{\tilde{n}} - S_{\tilde{m}}| < \varepsilon$

$\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq m \geq N : \ |S_n - S_{m-1}| < \varepsilon$

**Example:** $\sum_{k=1}^{\infty} (-1)^k$ Calculate: $\left| \sum_{k=N}^{N+2} (-1)^k \right| = \begin{cases} |1 + (-1) + 1| \\ |-1 + 1 + (-1)| \end{cases} = 1$

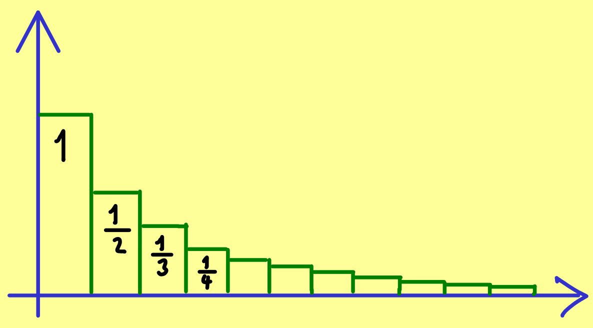
**Important fact:** $\sum_{k=1}^{\infty} a_k$ is convergent $\Longrightarrow$ $(a_k)_{k \in \mathbb{N}}$ convergent with $\lim_{k \to \infty} a_k = 0$

# Real Analysis – Part 18

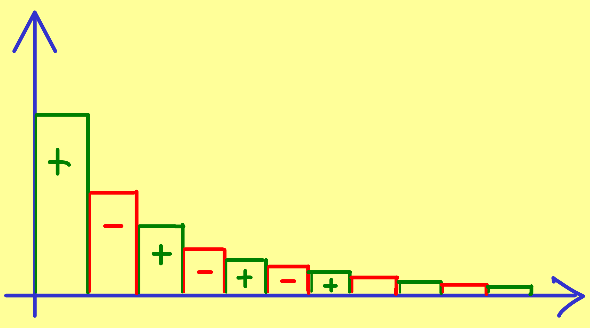
**Harmonic series:**

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

divergent




**Leibniz criterion:**




$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$$

convergent

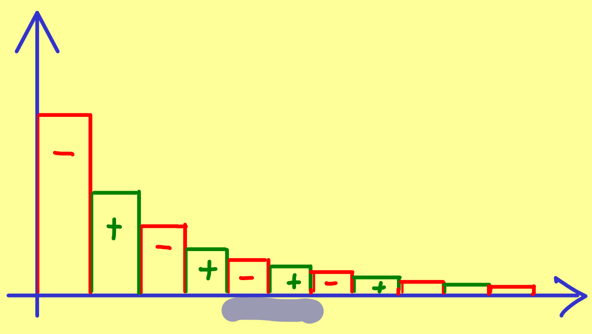
**Theorem:** (Alternating series test, Leibniz criterion, Leibniz's test)

Let $(a_k)_{k\in\mathbb{N}}$ be convergent with $\lim_{k\to\infty} a_k = 0$ and monotonically decreasing.

Then: $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent.

**Proof:** $S_n = \sum_{k=1}^{n} (-1)^k a_k$

$\Rightarrow \; a_k \geq 0$

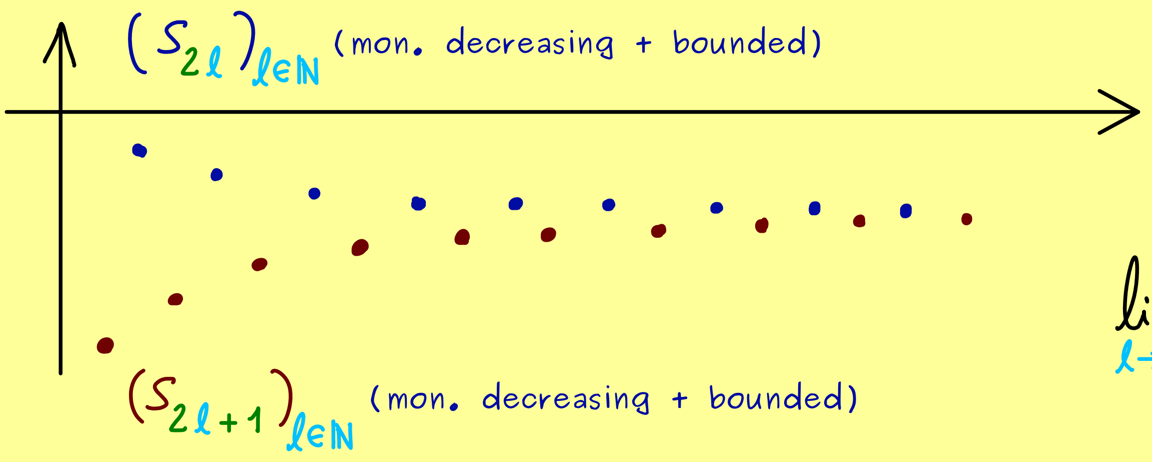



$$S_{2\ell+2} - S_{2\ell} = -a_{2\ell+1} + a_{2\ell+2} \leq 0 \qquad \text{(monotonically decreasing)}$$

$$S_{2\ell+3} - S_{2\ell+1} = a_{2\ell+2} - a_{2\ell+3} \geq 0 \qquad \text{(monotonically increasing)}$$

$$S_{2\ell+1} - S_{2\ell} = -a_{2\ell+1} \leq 0 \quad \Rightarrow \quad S_3 \leq S_{2\ell+1} \leq S_{2\ell} \leq S_2$$

(bounded)




$$\lim_{\ell\to\infty} \left(S_{2\ell+1} - S_{2\ell}\right) = \lim_{\ell\to\infty} \left(-a_{2\ell+1}\right) = 0$$

$$S := \lim_{\ell\to\infty} S_{2\ell+1} = \lim_{\ell\to\infty} S_{2\ell} \quad \Rightarrow \quad \lim_{n\to\infty} S_n = S \quad \text{(convergent!)}$$

**Example:** $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ convergent by Leibniz criterion

# Real Analysis – Part 19

$\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

abs. convergent $\Rightarrow$ convergent: $\sum_{k=1}^{\infty} |a_k|$ is convergent $\Rightarrow$

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq m \geq N : \; \sum_{k=m}^{n} |a_k| < \varepsilon \qquad \text{(Cauchy criterion)}$$

$$\Rightarrow \forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq m \geq N : \; \left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\overset{\text{(Cauchy criterion)}}{\Rightarrow} \sum_{k=1}^{\infty} a_k \text{ is convergent}$$

Counterexample: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is convergent but not absolutely convergent
(Leibniz criterion) (harmonic series)

Majorant criterion: Let $\sum_{k=1}^{\infty} a_k$ be a series.

If there is $n_0 \in \mathbb{N}$ and a convergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and with $|a_k| \leq b_k$ for all $k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is abs. convergent.

Proof: Apply Cauchy criterion to $\sum_{k=1}^{\infty} b_k$:

$$\forall \varepsilon > 0 \;\; \exists N \geq n_0 \;\; \forall n \geq m \geq N : \; \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

Minorant criterion: Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$.

If there is $n_0 \in \mathbb{N}$ and a divergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ and with $a_k \geq b_k$ for all $k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Example: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent because $\sqrt{k} \leq k \Leftrightarrow \frac{1}{\sqrt{k}} \geq \frac{1}{k}$ for all $k \geq 1$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent

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## Real Analysis – Part 20

$\sum_{k=1}^{\infty} a_k$ absolutely convergent?

There is a convergent majorant $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

We know the geometric series!

$$\sum_{k=0}^{\infty} q^k \text{ convergent} \iff |q|<1$$

Fact: If there is $n_0 \in \mathbb{N}$ and $C, q \in \mathbb{R}$ with $|q|<1$ such that $|a_k| \leq C \cdot q^k$ for all $k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Ratio test: If there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$$a_k \neq 0 \quad \text{and} \quad \left|\frac{a_{k+1}}{a_k}\right| \leq q \quad \text{for all } k \geq n_0,$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Proof: $$|a_{k+1}| \leq q \cdot |a_k| \leq q \cdot q \cdot |a_{k-1}| \leq \overset{\text{inductively}}{\cdots} \leq q^{k+1-n_0} |a_{n_0}| = q^{k+1} \overbrace{\frac{|a_{n_0}|}{q^{n_0}}}^{C}$$

Example: $\sum_{k=1}^{\infty} \frac{1}{k!}$ convergent? $\Rightarrow$ Yes! (by ratio test)

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{1}{k+1} \leq \underbrace{\frac{1}{2}}_{q} \quad \text{for all } k \geq \underbrace{1}_{n_0}.$$

$\boxed{(k+1)! = (k+1)\cdot k!}$

Warning: $\left|\frac{a_{k+1}}{a_k}\right| < 1$ is not enough!

Root test: If there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$$\sqrt[k]{|a_k|} \leq q \quad \text{for all } k \geq n_0,$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent!

Proof: $$\sqrt[k]{|a_k|} \leq q \iff |a_k| \leq q^k$$

Example: $\sum_{k=1}^{\infty} \left(\frac{3}{\sqrt{2+k}}\right)^{2k}$ convergent? Yes, by root test!

$$\sqrt[k]{\left(\frac{3}{\sqrt{2+k}}\right)^{2k}} = \left(\frac{3}{\sqrt{2+k}}\right)^2 = \frac{9}{2+k} \leq \frac{9}{10} \quad \text{for all } k \geq 8$$

Remember:

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is abs. convergent!}$$

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent!}$$

Attention: For the ratio test, this is different:

$$\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is abs. convergent!}$$

$$\liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent!}$$

(Remember: the ratio test is weaker than the root test, in general)

The Bright Side of Mathematics

# Real Analysis – Part 21

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n \qquad (n \text{ is even})$$
$$= a_2 + a_1 + a_4 + a_3 + \cdots + a_n + a_{n-1}$$

Reordering does not change a finite sum!

**Example:** $\sum_{k=0}^{\infty} (-1)^k = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

- is not convergent
- but has two accumulation values $0, 1$

a reordering $= 1 + 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

- is not convergent
- but has two accumulation values $1, 2$

**Example:** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ convergent by the Leibniz criterion

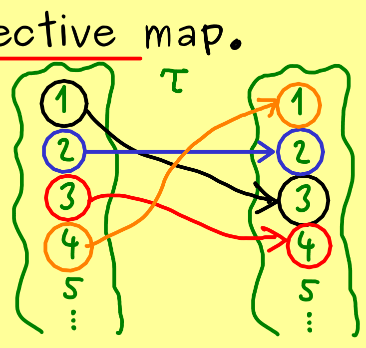
$$= \frac{1}{1} + \left(-\frac{1}{2}\right) + \frac{1}{3} + \left(-\frac{1}{4}\right) + \frac{1}{5} + \left(-\frac{1}{6}\right) + \frac{1}{7} + \cdots = \overset{=\log(2)}{C} > 0$$

a reordering $= \frac{1}{1} + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \frac{1}{9} + \frac{1}{11} + \left(-\frac{1}{6}\right) + \cdots$

$= \frac{3}{2} \cdot C$ different limits!

**Definition:** Let $\sum_{k=1}^{\infty} a_k$ be a series and $\tau : \mathbb{N} \to \mathbb{N}$ be a bijective map.

Then $\sum_{k=1}^{\infty} a_{\tau(k)}$ is called a reordering of $\sum_{k=1}^{\infty} a_k$ .




**Theorem:** If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then: (for any $\tau : \mathbb{N} \to \mathbb{N}$ bijective)

$\sum_{k=1}^{\infty} a_{\tau(k)}$ is also abs. convergent and $\sum_{k=1}^{\infty} a_{\tau(k)} = \sum_{k=1}^{\infty} a_k$ .

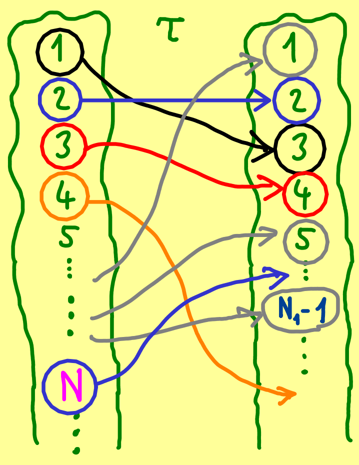
**Proof:** Let $\varepsilon > 0$. Cauchy criterion $\Rightarrow$ $\exists N_1 \in \mathbb{N} \; \forall n \geq m \geq N_1 :$ $\sum_{k=m}^{n} |a_k| < \varepsilon$

$$\Bigg| \overbrace{\sum_{k=1}^{\infty} a_k}^{A} - \sum_{k=1}^{n} a_{\tau(k)} \Bigg| = \Bigg| A - \sum_{k=1}^{N_1-1} a_k + \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^{n} a_{\tau(k)} \Bigg|$$

$$\leq \underbrace{\Bigg| A - \sum_{k=1}^{N_1-1} a_k \Bigg|} + \underbrace{\Bigg| \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^{n} a_{\tau(k)} \Bigg|}_{=(*)} < \underbrace{2 \cdot \varepsilon}_{\varepsilon'}$$

$$= \Bigg| \sum_{k=N_1}^{\infty} a_k \Bigg| = \lim_{n \to \infty} \Bigg| \sum_{k=N_1}^{n} a_k \Bigg|$$

$$\leq \lim_{n \to \infty} \sum_{k=N_1}^{n} |a_k| < \varepsilon$$




For $n \geq N$ : $\{ \tau(1), \tau(2), \ldots, \tau(n) \} \supseteq \{ 1, 2, \ldots, N_1 - 1 \}$

$$\Rightarrow \; (*) = \Bigg| \sum_{k=?}^{?} a_{\tau(k)} \Bigg| \leq \sum_{k=?}^{?} |a_{\tau(k)}| \leq \sum_{j=N_1}^{\infty} |a_j| < \varepsilon$$

$$\Rightarrow \; \forall \varepsilon' > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N : \quad \Bigg| \overbrace{\sum_{k=1}^{\infty} a_k}^{A} - \sum_{k=1}^{n} a_{\tau(k)} \Bigg| < \varepsilon'$$

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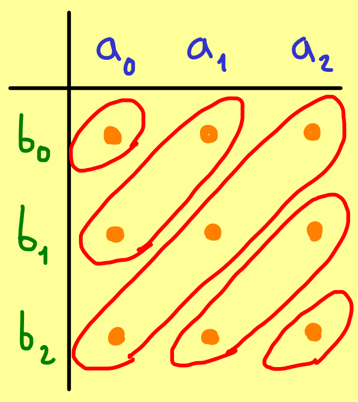
## Real Analysis – Part 22

$$\sum_{k=0}^{\infty} a_k \,,\ \sum_{k=0}^{\infty} b_k \quad \xrightarrow{\text{How to multiply?}} \quad \sum_{k=0}^{\infty} c_k$$

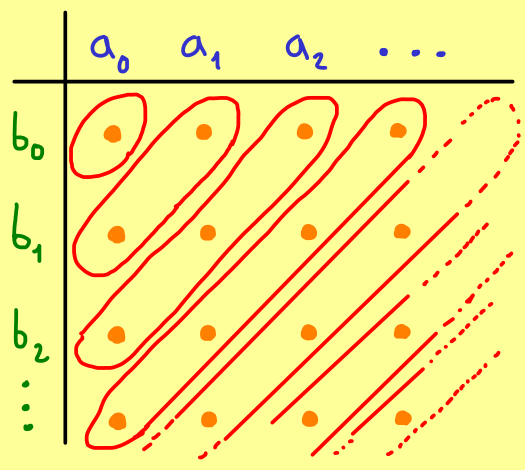
For finite sums: $(a_0 + a_1 + a_2)\cdot(b_0 + b_1 + b_2)$

$$= a_0 b_0 + a_1 b_0 + a_2 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_1 + a_0 b_2 + a_1 b_2 + a_2 b_2$$

$$= \underbrace{(a_0 b_0)}_{0} + \underbrace{(a_1 b_0 + a_0 b_1)}_{1} + \underbrace{(a_2 b_0 + a_1 b_1 + a_0 b_2)}_{2} + \underbrace{(a_2 b_1 + a_1 b_2)}_{3} + \underbrace{(a_2 b_2)}_{4}$$




Cauchy product: For two series $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$, the series $\sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{\ell=0}^{k} a_\ell \, b_{k-\ell}$ is called the Cauchy product.




Theorem: If $\sum_{k=0}^{\infty} a_k$ is absolutely convergent and $\sum_{k=0}^{\infty} b_k$ convergent, then Cauchy product $\sum_{k=0}^{\infty} c_k$ is abs. convergent and $\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\cdot\left(\sum_{k=0}^{\infty} b_k\right)$

Example: $\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$ (abs. convergent by the ratio test)

Apply Cauchy product for $\exp(x)$ and $\exp(y)$ :

$$c_k = \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!}\cdot\frac{y^{k-\ell}}{(k-\ell)!} = \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell} x^\ell y^{k-\ell}$$

binomial coefficient: $\binom{k}{\ell} = \frac{k!}{\ell!\,(k-\ell)!}$

$$\overset{\text{binomial theorem}}{=} \frac{1}{k!}(x+y)^k$$

$$\exp(x+y) = \sum_{k=0}^{\infty}\frac{1}{k!}(x+y)^k = \sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \exp(x)\cdot\exp(y)$$

$$\Longrightarrow \quad \boxed{\exp(x+y) = \exp(x)\cdot\exp(y)}$$

fundamental multiplicative identity

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ON STEADY

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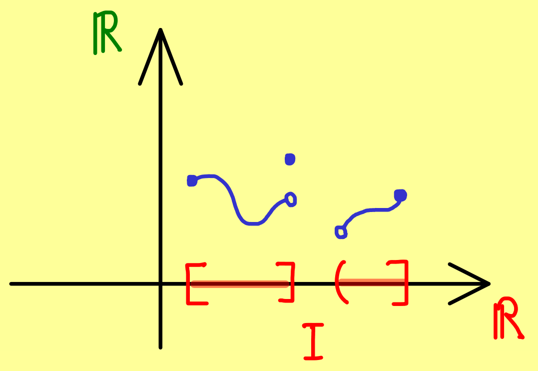
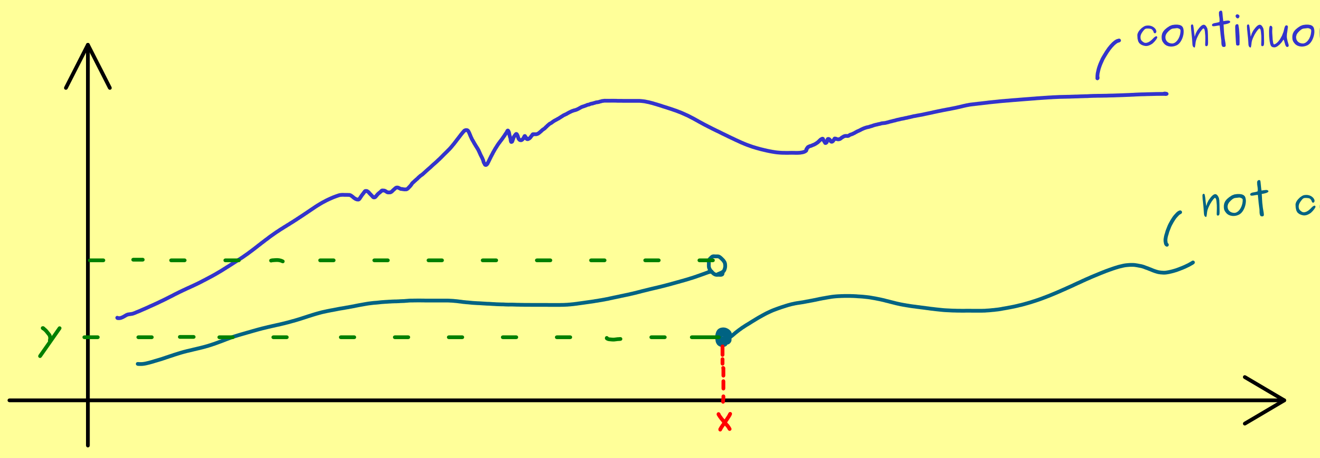
## Real Analysis – Part 23

**Function:** $f: I \to \mathbb{R}$ $\quad (I \subseteq \mathbb{R})$




**Later:** continuous functions $f: \mathbb{R} \to \mathbb{R}$



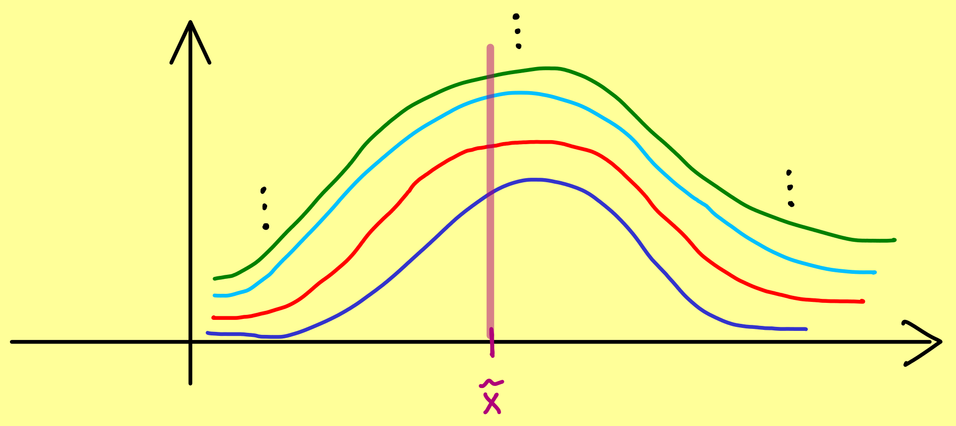

Idea: small errors on x-axis $\rightsquigarrow$ small errors on y-axis

**Definition:** $f: I \to \mathbb{R}$ is called a bounded function if

$$\{ f(x) \mid x \in I \} = \mathrm{Ran}(f) = f[I] \text{ is a bounded set in } \mathbb{R}$$

$$\left( \Longleftrightarrow \sup_{x \in I} |f(x)| < \infty \right)$$

**Sequence of functions:**




sequence:

$(f_1, f_2, f_3, f_4, f_5, \dots)$

with sequence members:

$f_1: I \to \mathbb{R}$

$f_2: I \to \mathbb{R}$

$f_3: I \to \mathbb{R}$

$\vdots$

For any fixed $\tilde{x} \in I$,
we get an ordinary sequence of real numbers:

$$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), f_4(\tilde{x}), f_5(\tilde{x}), \dots)$$

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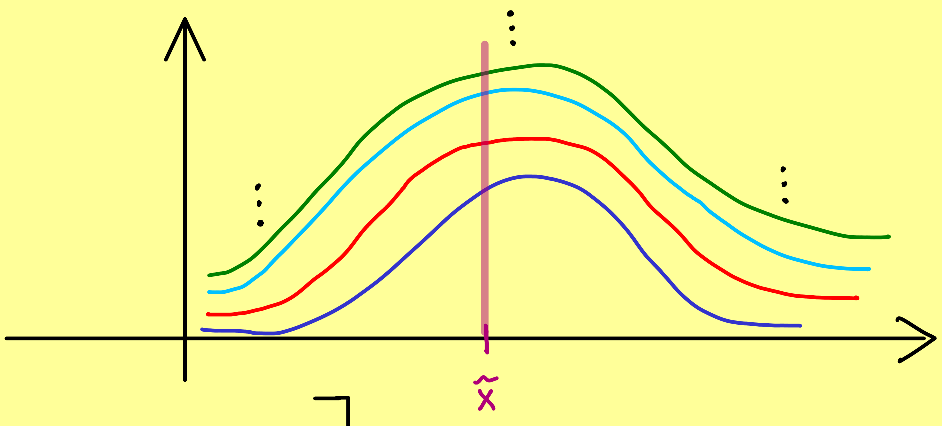
# Real Analysis – Part 24

sequence of functions: $(f_1, f_2, f_3, f_4, f_5, \dots)$ $\quad f_n : I \longrightarrow \mathbb{R}$

Pointwise convergence: $(f_1, f_2, f_3, f_4, f_5, \dots)$ is pointwisely convergent to a function $f : I \longrightarrow \mathbb{R}$ if for all $\tilde{x} \in I$:

$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), f_4(\tilde{x}), f_5(\tilde{x}), \dots)$

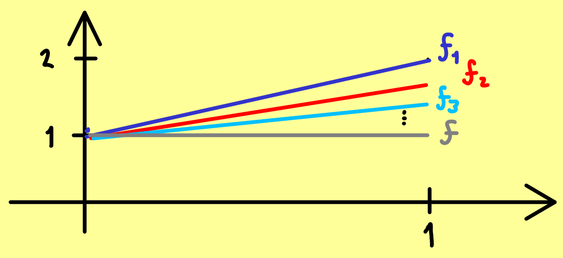
is convergent to $f(\tilde{x})$.




$$\left[ \forall \tilde{x} \in I \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon \right]$$

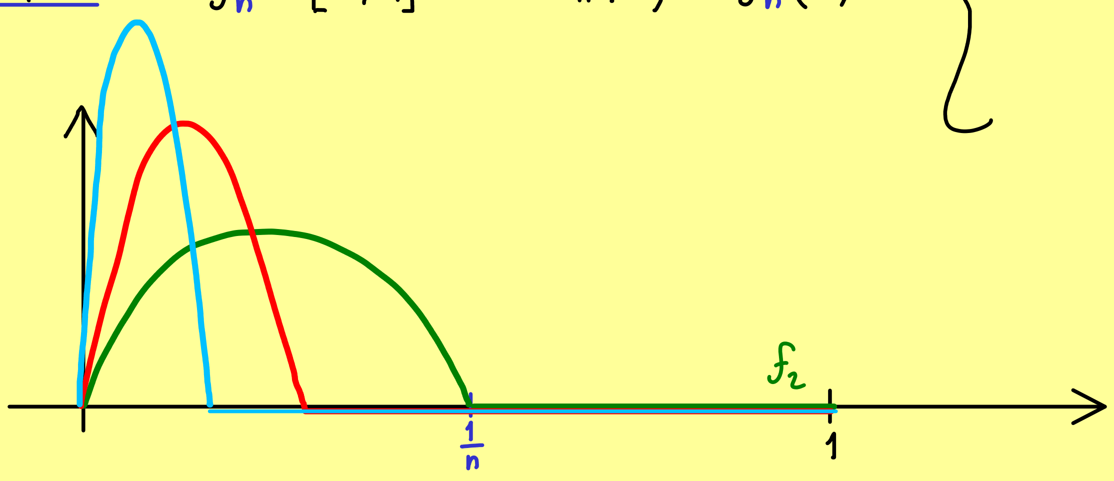
Example: $f_n : [0,1] \longrightarrow \mathbb{R}, \quad f_n(x) = \frac{1}{n} x + 1$

For $\tilde{x} \in [0,1]$: $\quad f_n(\tilde{x}) = \frac{1}{n}\tilde{x} + 1 \xrightarrow{n \to \infty} 1$




$\Longrightarrow$ (pointwise) limit function $\quad f : [0,1] \longrightarrow \mathbb{R}, \quad f(x) = 1$

Example: $f_n : [0,1] \longrightarrow \mathbb{R}, \quad f_n(x) = \begin{cases} n^2 x (1 - nx) & , \; x \in [0, \frac{1}{n}] \\ 0 & , \; x \in (\frac{1}{n}, 1] \end{cases}$



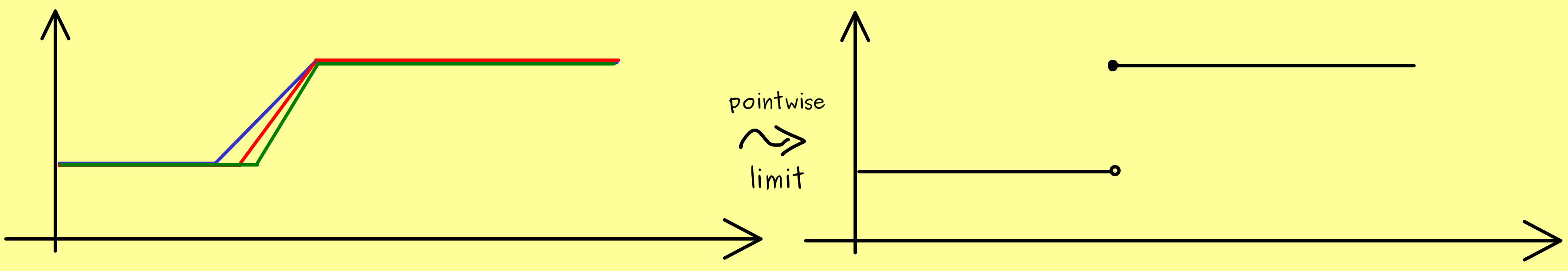

$$f_n\left(\frac{1}{2} \cdot \frac{1}{n}\right) = n^{\cancel{2}} \frac{1}{2\cancel{n}} \left(1 - \cancel{n} \frac{1}{2\cancel{n}}\right) = \frac{n}{4}$$

For $x = 0$: $\quad f_n(x) = 0 \quad$ for all $n \in \mathbb{N}$

For $x > 0$: $\quad f_n(x) = 0 \quad$ for all $n > \frac{1}{x}$

$\Longrightarrow$ (pointwise) limit function $\quad f : [0,1] \longrightarrow \mathbb{R}, \quad f(x) = 0$

Example:



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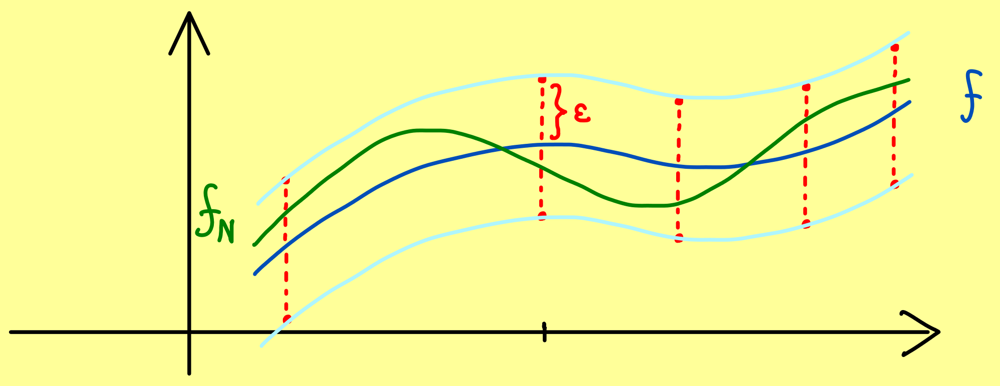
## Real Analysis – Part 25

$(f_1, f_2, f_3, f_4, f_5, \dots)$ is pointwisely convergent to $f: I \to \mathbb{R}$

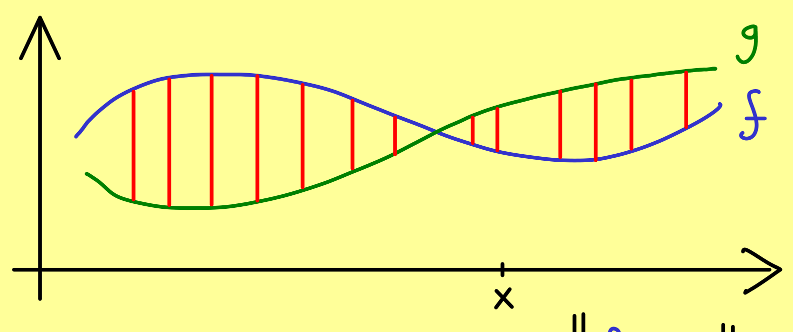
$$\forall \tilde{x} \in I \quad \forall \varepsilon > 0 \quad \exists N_{\tilde{x}} \in \mathbb{N} \quad \forall n \geq N: \quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon$$

Definition: $(f_1, f_2, f_3, f_4, f_5, \dots)$ is uniformly convergent to $f: I \to \mathbb{R}$ if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall \tilde{x} \in I: \quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon$$




Distance for functions:



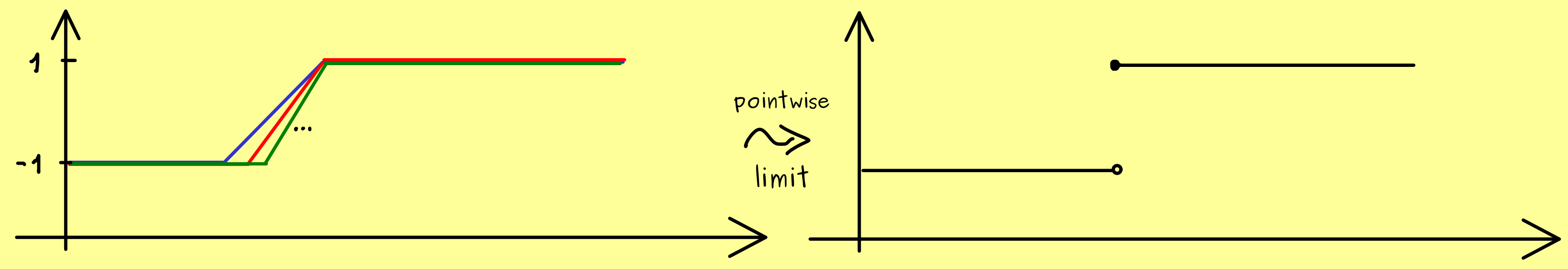

$f: I \to \mathbb{R}$

$g: I \to \mathbb{R}$

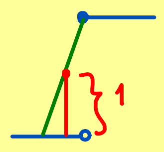
supremum norm of $f - g$ $\quad \rightsquigarrow \quad \|f - g\|_\infty = \sup_{x \in I} |f(x) - g(x)|$

Uniform convergence means: $\|f_n - f\|_\infty \xrightarrow{n \to \infty} 0$

Example:




$\|f_n - f\|_\infty \geq 1$ for all $n$




Result: pointwise convergence $\not\Rightarrow$ uniform convergence

$\Leftarrow$

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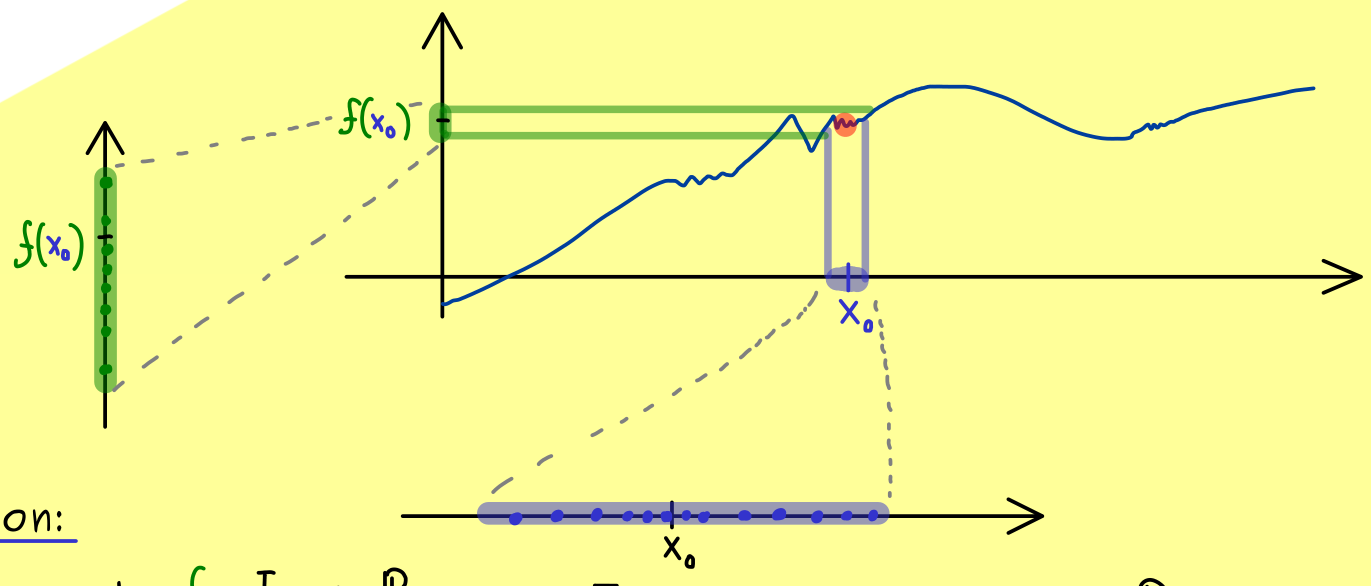
## Real Analysis – Part 26

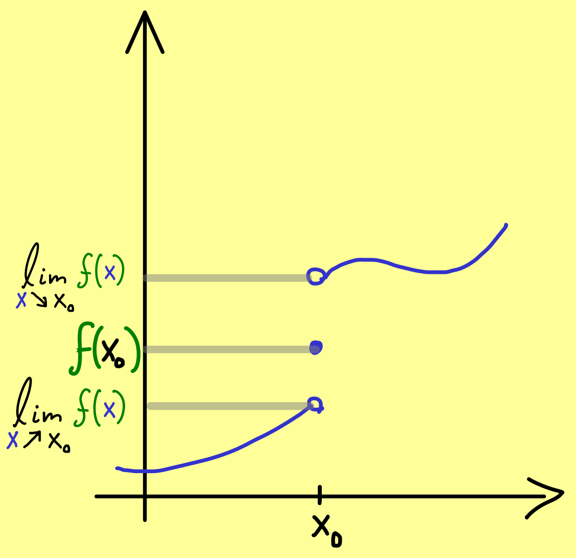



**Definition:** Let $f: I \to \mathbb{R}$, $x_0 \in I$. If there is $c \in \mathbb{R}$ and all sequences $(x_n)_{n\in\mathbb{N}} \subseteq I \setminus \{x_0\}$ with $\lim\limits_{n\to\infty} x_n = x_0$ we have $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is also convergent with $\lim\limits_{n\to\infty} f(x_n) = c$,

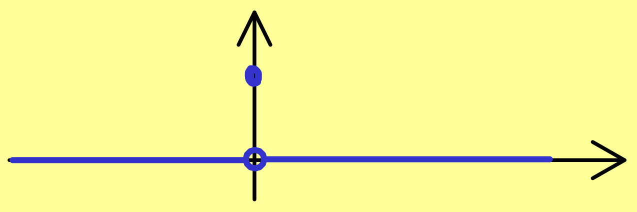
then we write $\boxed{\lim\limits_{x\to x_0} f(x) = c}$

and $\boxed{\lim\limits_{x\nearrow x_0} f(x) = c}$ if $x_n < x_0$ for all $n \in \mathbb{N}$

and $\boxed{\lim\limits_{x\searrow x_0} f(x) = c}$ if $x_n > x_0$ for all $n \in \mathbb{N}$




**Example:** (a) $f(x) = \begin{cases} 0 & , x \neq 0 \\ 1 & , x = 0 \end{cases}$



$$\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$$

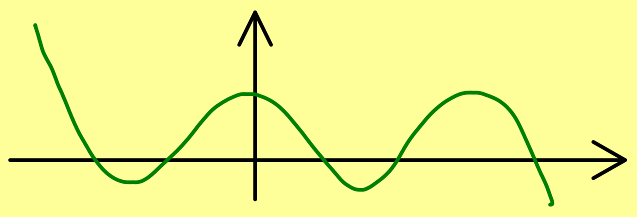
(b) $f(x) = a_m \cdot x^m + a_{m-1} x^{m-1} + \dots + a_1 \cdot x^1 + a_0$ $\quad (f: \mathbb{R} \to \mathbb{R})$

For $x_0 \in \mathbb{R}$ take $(x_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} x_n = x_0$

$$f(x_n) = a_m \cdot x_n^m + a_{m-1} x_n^{m-1} + \dots + a_1 \cdot x_n^1 + a_0$$

$$\xrightarrow[\text{(limit theorems)}]{n\to\infty} a_m \cdot x_0^m + a_{m-1} x_0^{m-1} + \dots + a_1 \cdot x_0^1 + a_0 = f(x_0)$$

$$\Rightarrow \lim_{x\to x_0} f(x) = f(x_0)$$



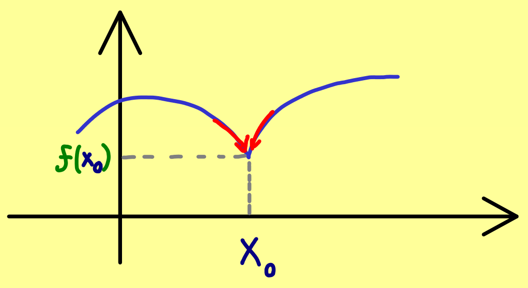
# The Bright Side of Mathematics

## Real Analysis – Part 27

<u>Definition:</u> Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

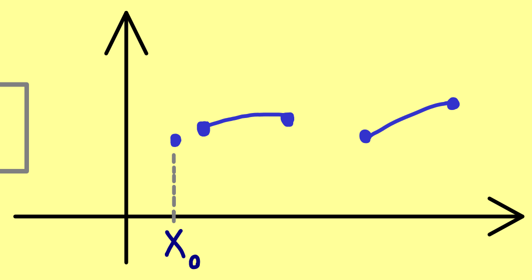
$f$ is called continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$




or if $x_0$ is isolated in $I$.

There is no sequence $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$

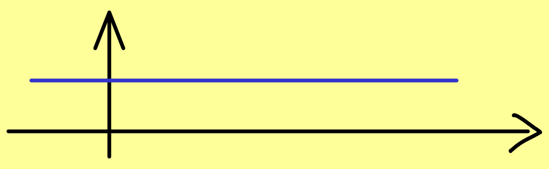



<u>Definition:</u> Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

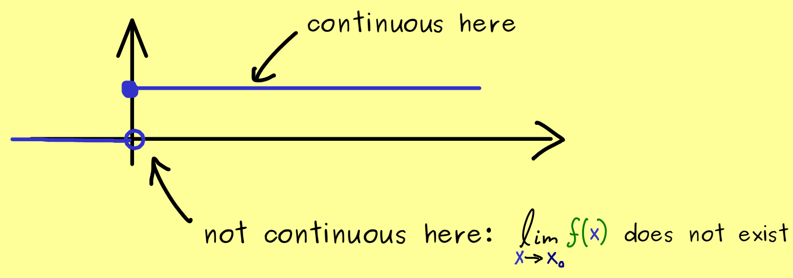
$f$ is called continuous (on $I$) if $f$ is continuous at $x_0$ for all $x_0 \in I$.

<u>To remember:</u> Continuity implies: $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$ (if $\lim_{n\to\infty} x_n \in I$)

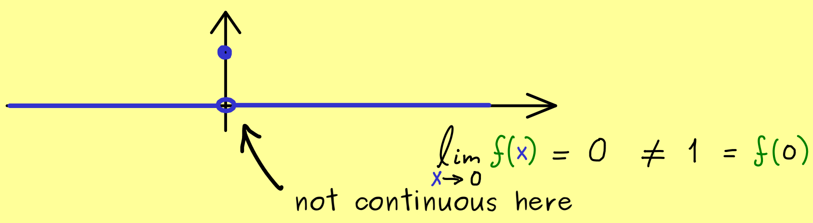
<u>Examples:</u> (a) $f: I \to \mathbb{R}$ constant



(b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$




(c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

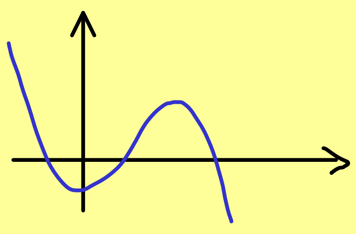



(d) $f: \mathbb{R} \to \mathbb{R}$ polynomial

$$f(x) = a_m \cdot x^m + a_{m-1} x^{m-1} + \cdots + a_1 \cdot x^1 + a_0$$

We have: $\lim_{x\to x_0} f(x) = f(x_0)$ for all $x_0 \in I$.

limit theorem for sequences



(e) $f: I \to \mathbb{R}$ rational function

$I := \{x \in \mathbb{R} \mid q(x) \neq 0\}$

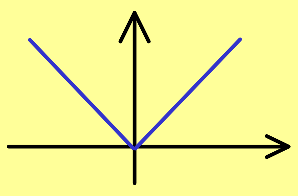
$$f(x) = \frac{p(x)}{q(x)}$$

polynomial

polynomial

continuous on $I$

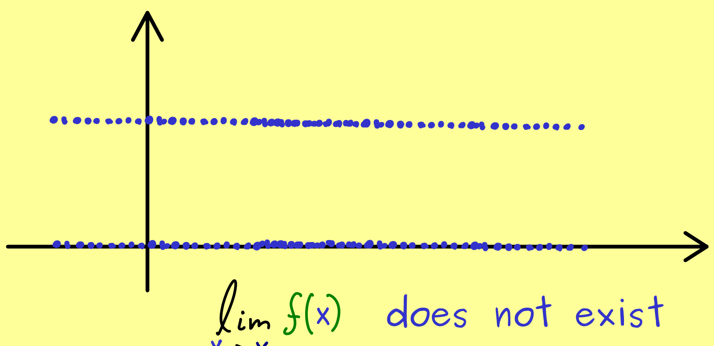
(f) $f: \mathbb{R} \to \mathbb{R}$ absolute value $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$



$$\left.\begin{aligned} \lim_{x \nearrow 0} f(x) &= \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (-x_n) = 0 \\ \lim_{x \searrow 0} f(x) &= \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n) = 0 \end{aligned}\right\} \Rightarrow \lim_{x\to x_0} f(x) = f(x_0)$$

(g) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$

($\mathbb{Q}$ is dense in $\mathbb{R}$ by construction)



$\lim_{x\to x_0} f(x)$ does not exist

The Bright Side of Mathematics

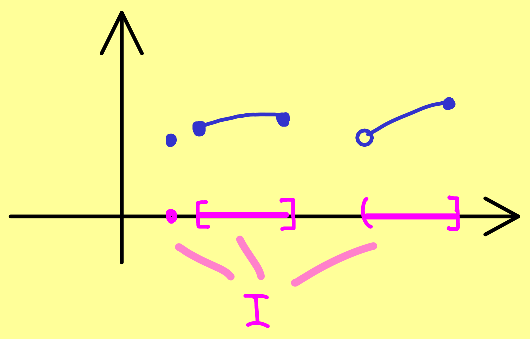
# Real Analysis – Part 28

**Continuity:** $f$ is called continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

**Theorem:** Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

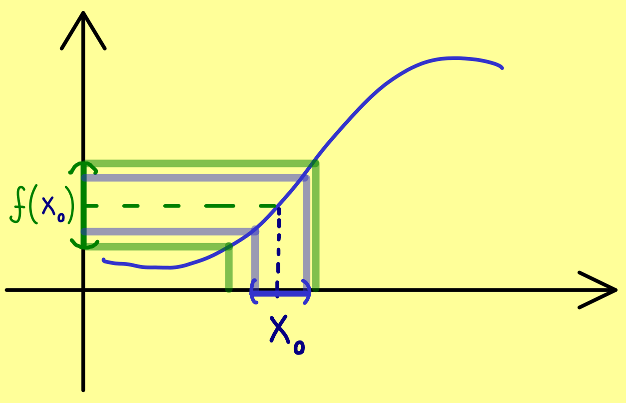
For $x_0 \in I$, we have:

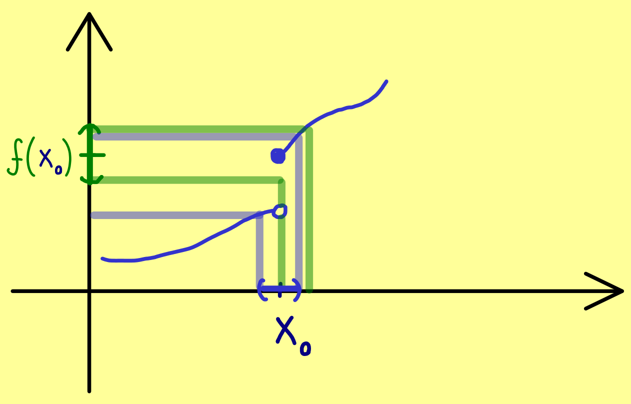



$f$ is continuous at $x_0 \in I$

$$\Longleftrightarrow$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I : \quad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon$$




**Proof:** ($\Longrightarrow$) Assume $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in I : \ |x - x_0| < \delta \ \wedge \ |f(x) - f(x_0)| \geq \varepsilon$

Take $\frac{1}{n}$, $n \in \mathbb{N}$

$\Longrightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in I \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$ $\Longrightarrow$ $f$ is not continuous at $x_0 \in I$

($\Longleftarrow$) Choose sequence $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with limit $x_0$. Let $\varepsilon > 0$. Take $\delta > 0$. (from assumption)

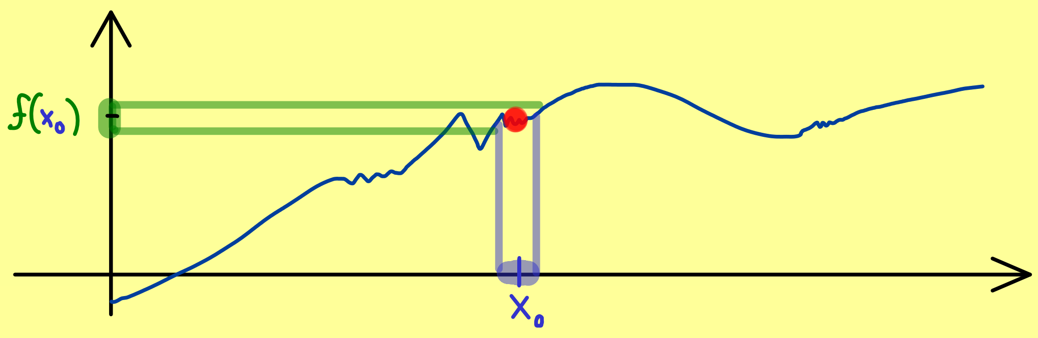
There is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - x_0| < \delta$.

Also (by assumption) we have $|f(x_n) - f(x_0)| < \varepsilon$. $\Longrightarrow$ $f$ is continuous at $x_0 \in I$

$\square$

# The Bright Side of Mathematics

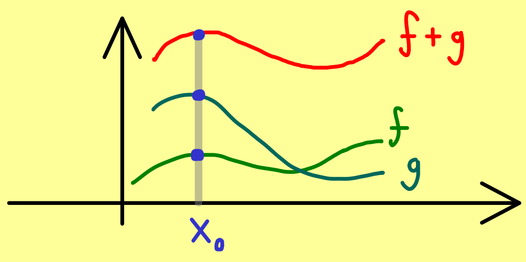
## Real Analysis – Part 29




**Proposition:** $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ continuous at $x_0 \in I$,

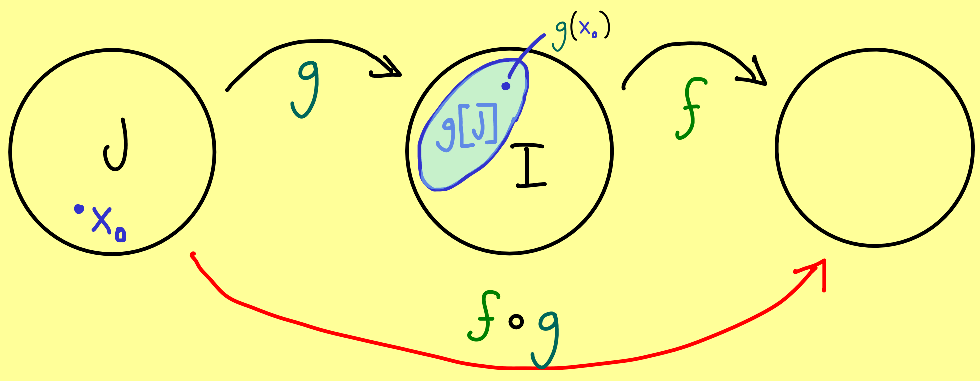
then $f + g: I \to \mathbb{R}$ continuous at $x_0 \in I$,

$f \cdot g: I \to \mathbb{R}$ continuous at $x_0 \in I$.




If in addition $g(x_0) \neq 0$, then $\dfrac{f}{g}$ is continuous at $x_0 \in I$.

**Composition of functions:**




**Proposition:** $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$, $I, J \subseteq \mathbb{R}$, with $g[J] \subseteq I$.

$$\left.\begin{array}{l} g \text{ continuous at } x_0 \in J \\ f \text{ continuous at } g(x_0) \in I \end{array}\right\} \Longrightarrow f \circ g : J \to \mathbb{R} \text{ continuous at } x_0 \in J$$

**Proof:** Choose sequence $(x_n)_{n\in\mathbb{N}} \subseteq J \setminus \{x_0\}$ with limit $x_0$.

$\boxed{f \text{ is continuos at } g(x_0) \text{ and } \lim\limits_{n\to\infty} g(x_n) = g(x_0)}$

$$\lim_{n\to\infty} (f \circ g)(x_n) = \lim_{n\to\infty} f(g(x_n)) = f\left(\lim_{n\to\infty} g(x_n)\right)$$

$\boxed{g \text{ is continuous at } x_0}$

$$= f\left(g\left(\lim_{n\to\infty} x_n\right)\right) = (f \circ g)(x_0)$$

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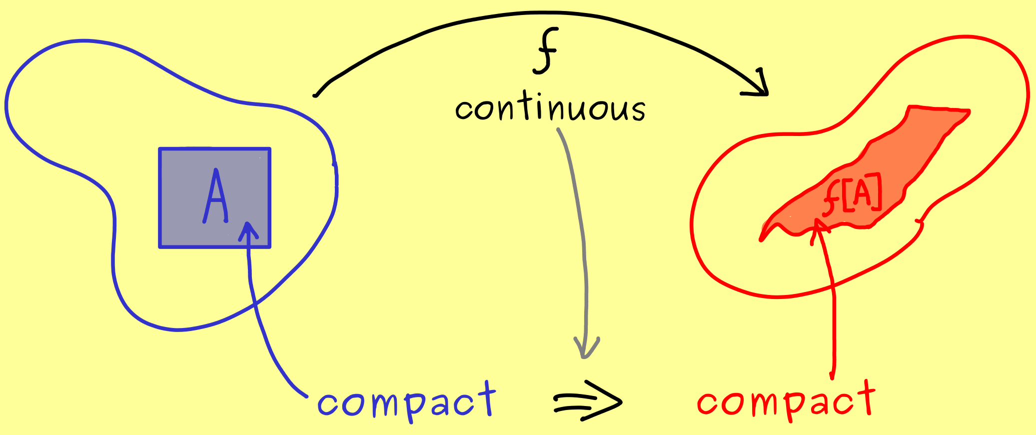
# Real Analysis – Part 30



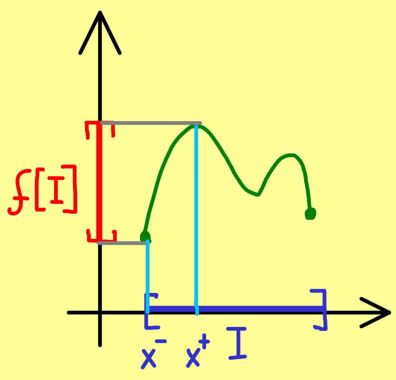

**Theorem:** $I \subseteq \mathbb{R}$ compact, $f: I \to \mathbb{R}$ continuous.

Then: $f[I] \subseteq \mathbb{R}$ is compact ($\overset{\text{Heine–Borel}}{=}$ bounded + closed)

and there are $x^+, x^- \in I$ with

$$f(x^+) = \sup\{ f(x) \mid x \in I \}$$

$$f(x^-) = \inf\{ f(x) \mid x \in I \}$$




**Proof:** Compact means: every sequence has a convergent subsequence.

Let $(y_n)_{n\in\mathbb{N}} \subseteq f[I]$ be a sequence.

For each $y_n$ there is $x_n \in I$ with $f(x_n) = y_n$. $\Rightarrow$ New sequence $(x_n)_{n\in\mathbb{N}} \subseteq I$

$\overset{I \text{ compact}}{\Longrightarrow}$ There is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent: $x := \lim_{k\to\infty} x_{n_k} \in I$

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \underset{\uparrow\, f \text{ continuous}}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x) =: y$$

So $(y_{n_k})_{k\in\mathbb{N}}$ is convergent with limit $y \in f[I]$. $\Rightarrow$ $f[I]$ compact

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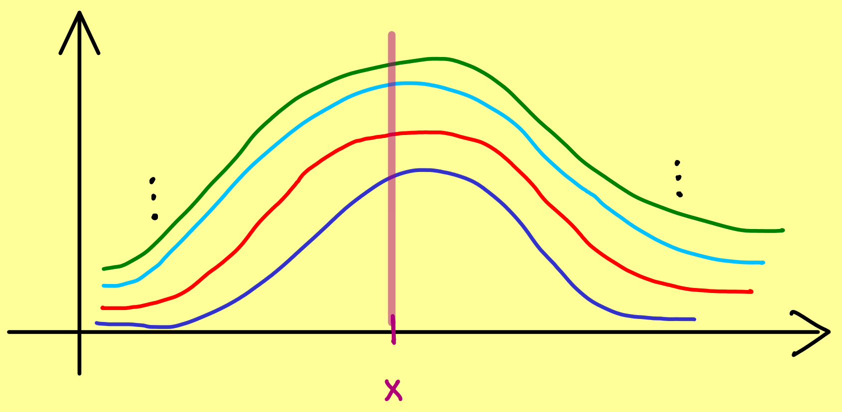
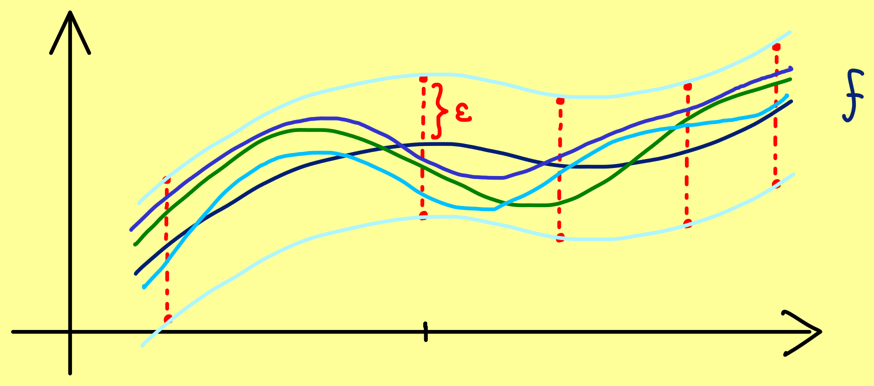
## Real Analysis – Part 31

$(f_1, f_2, f_3, f_4, f_5, \dots)$

pointwise convergence:




uniform convergence:

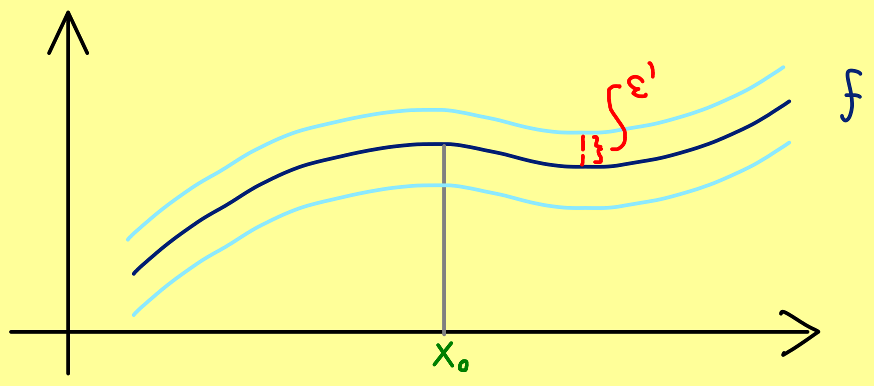



**Theorem:** $I \subseteq \mathbb{R}$, $f_n : I \longrightarrow \mathbb{R}$ continuous (for all $n \in \mathbb{N}$), and

$(f_n)_{n\in\mathbb{N}}$ uniformly converges to $f : I \longrightarrow \mathbb{R}$.

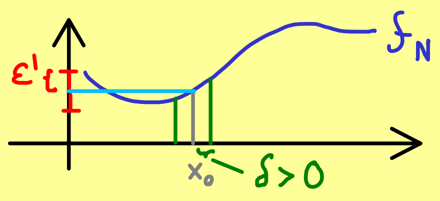
Then: $f$ is also continuous.

**Proof:** Let $\varepsilon > 0$. Let $x_0 \in I$. Set: $\varepsilon' := \frac{\varepsilon}{3}$ (see end of the proof)




Uniform convergence: $\forall \varepsilon' > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N \;\; \forall \tilde{x} \in I : \; |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon'$

Continuity of $f_N$:




We find $\delta > 0$ with:

$\forall x \in I : \; |x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \varepsilon'$

Hence: $|f(x) - f(x_0)| = |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)|$

$$\leq \underbrace{|f(x) - f_N(x)|}_{< \varepsilon'} + \underbrace{|f_N(x) - f_N(x_0)|}_{< \varepsilon'} + \underbrace{|f_N(x_0) - f(x_0)|}_{< \varepsilon'}$$

$$< 3 \cdot \varepsilon' = \varepsilon$$

Conclusion: We find $\delta > 0$ with: $\forall x \in I : \; |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

$\implies f$ is continuous at $x_0$ $\xRightarrow{x_0 \text{ arbitrary}}$ $f$ is continuous $\square$

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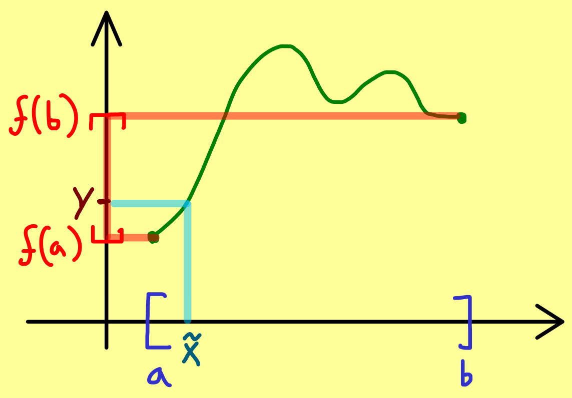
# Real Analysis – Part 32

$f: \underset{\substack{\| \\ [a,b]}}{I} \longrightarrow \mathbb{R}$ continuous



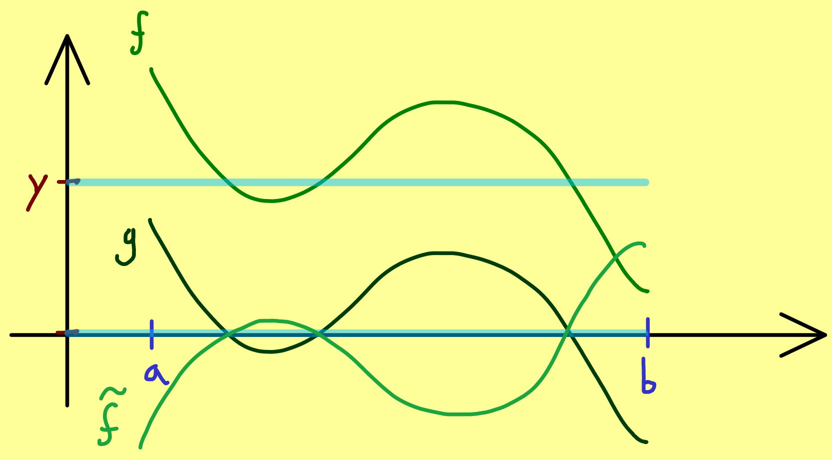

**Intermediate value theorem:** Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and

$y \in [f(a), f(b)]$ or $y \in [f(b), f(a)]$.

Then there is $\tilde{x} \in [a,b]$ with $f(\tilde{x}) = y$.

**Corollary:** $f\big[[a,b]\big]$ is also an interval.

**Proof of the intermediate value theorem:**

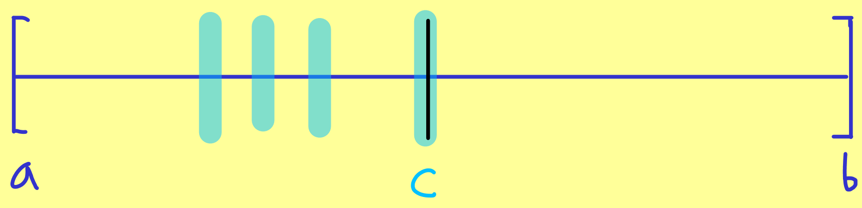



Define new function:

$g := f - y$

$$\tilde{f} := \begin{cases} -g & \text{if } g(a) > 0 \\ g & \text{if } g(a) \leq 0 \end{cases}$$

Then $\tilde{f}$ is continuous, $\tilde{y} := 0$, and $\tilde{f}(a) \leq 0$, $\tilde{f}(b) \geq 0$.




Check $\tilde{f}(c)$:

Repeat

If $\tilde{f}(c) > 0$ : $b_1 := c$, $a_1 := a$

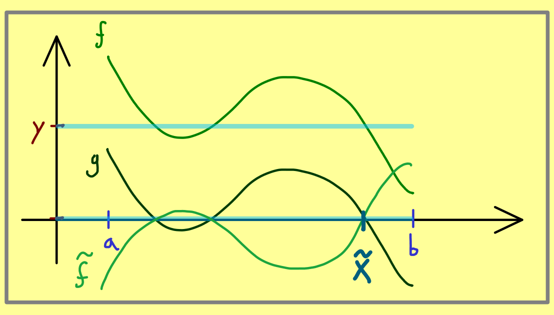
If $\tilde{f}(c) \leq 0$ : $a_1 := c$, $b_1 := b$

Wet get two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $b_n - a_n \xrightarrow{n\to\infty} 0$

$$\Longrightarrow \tilde{x} := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

We know:

$$\begin{array}{l} \lim\limits_{n\to\infty} \tilde{f}(a_n) \leq 0 \\ \lim\limits_{n\to\infty} \tilde{f}(b_n) \geq 0 \end{array} \Longrightarrow \begin{array}{l} \tilde{f}\big(\lim\limits_{n\to\infty} a_n\big) \leq 0 \\ \tilde{f}\big(\lim\limits_{n\to\infty} b_n\big) \geq 0 \end{array} \Longrightarrow \begin{array}{l} \tilde{f}(\tilde{x}) \leq 0 \\ \tilde{f}(\tilde{x}) \geq 0 \end{array}$$




$$\Longrightarrow \tilde{f}(\tilde{x}) = 0 \Longrightarrow \underset{\substack{\| \\ f(\tilde{x}) - y}}{g(\tilde{x})} = 0 \Longrightarrow f(\tilde{x}) = y \qquad \square$$

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# Real Analysis – Part 33

## ① Exponential function

$\exp: \mathbb{R} \to \mathbb{R}$ defined by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$e := \exp(1)$ Euler's number

$$\| \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718\ldots$$

We have shown: $\exp(x+y) = \exp(x)\cdot\exp(y)$

For example: $\exp(2) = \exp(1+1) = \exp(1)\cdot\exp(1) = e^2$

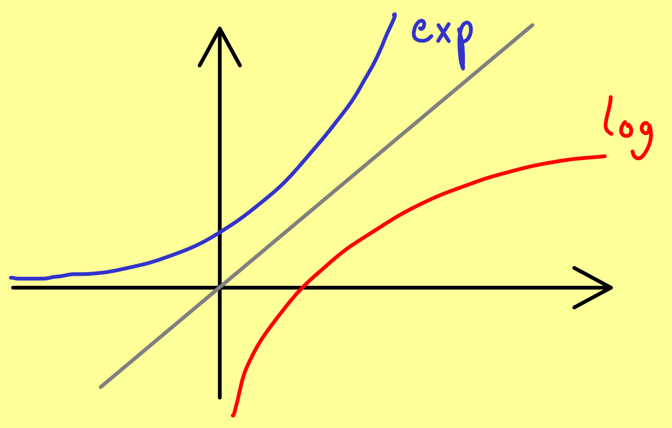
In general: $\exp(x) = e^x$ for $x \in \mathbb{R}$

More properties:

- $\exp$ is a continuous function
- $\exp$ is strictly monotonically increasing $\left( x<y \Rightarrow \exp(x) < \exp(y) \right)$
- $\lim\limits_{x\to\infty} \exp(x) = \infty$ , $\lim\limits_{x\to-\infty} \exp(x) = 0$
- $\exp: \mathbb{R} \to (0,\infty)$ is bijective

## ② Logarithm function

$\log: (0,\infty) \to \mathbb{R}$ defined by the inverse of $\exp: \mathbb{R} \to (0,\infty)$




- $\log$ is a continuous function
- $\log$ is strictly monotonically increasing
- $\log(x\cdot y) = \log(x) + \log(y)$

## ③ Polynomials

$f: \mathbb{R} \to \mathbb{R}$ , $f(x) = \underbrace{a_m}_{\neq 0} \cdot x^m + a_{m-1} x^{m-1} + \cdots + a_1 \cdot x^1 + a_0$ polynomial has degree $m$

continuous

## ④ Power series

$f: D \to \mathbb{R}$ , $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$ , $D := \left\{ x\in\mathbb{R} \;\middle|\; \sum_{k=0}^{\infty} a_k\cdot x^k \text{ converges} \right\}$

Example: $(a_k) = \left(0, \frac{1}{1!}, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, 0, \frac{1}{9!}, 0, -\frac{1}{11!}, \ldots\right)$

gives power series $\sin(x) := \sum_{k=0}^{\infty} a_k \cdot x^k$

**Theorem:** For a power series $\sum_{k=0}^{\infty} a_k\cdot x^k$ , there is a maximal $r \in [0,\infty) \cup \{\infty\}$ with $(-r, r) \subseteq D$. It holds: $\limsup\limits_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{r}$ $\left( \frac{1}{0} = \infty \quad \frac{1}{\infty} = 0 \right)$

power series is continuous on this interval

(Cauchy–Hadamard)

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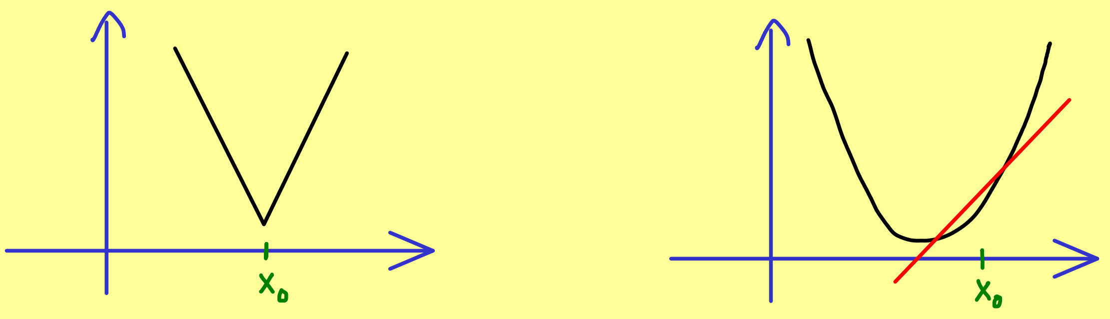
## Real Analysis – Part 34

Differentiability  (linearisation)  smoothness

$f: \mathbb{R} \to \mathbb{R}$




slope at point $x_0$ ?

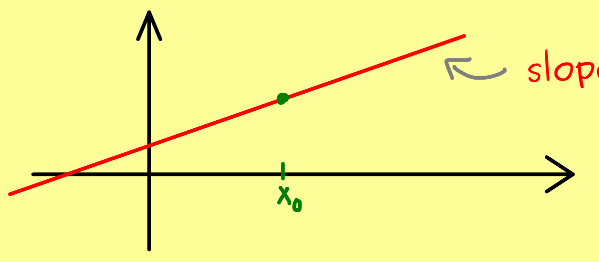
approximate $f$ locally with a linear function?

(affine) linear function: (linear polynomial)

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = a_1 \cdot x + a_0 = m \cdot (x - x_0) + c$

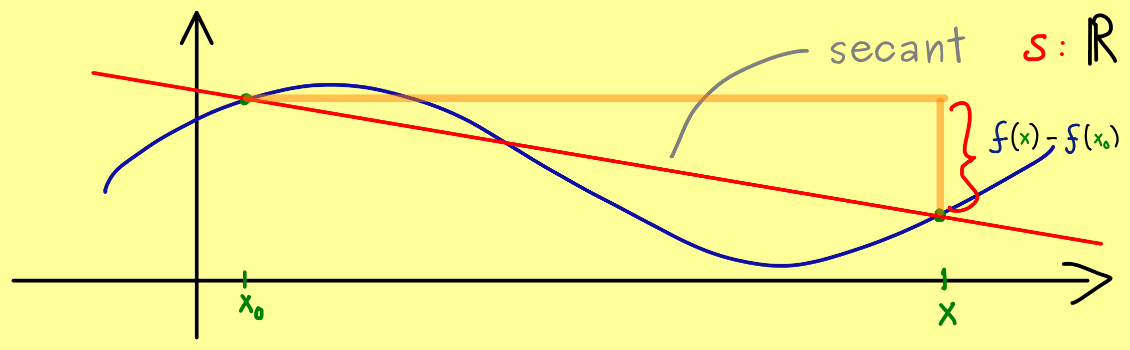
slope

constant $c = g(x_0)$



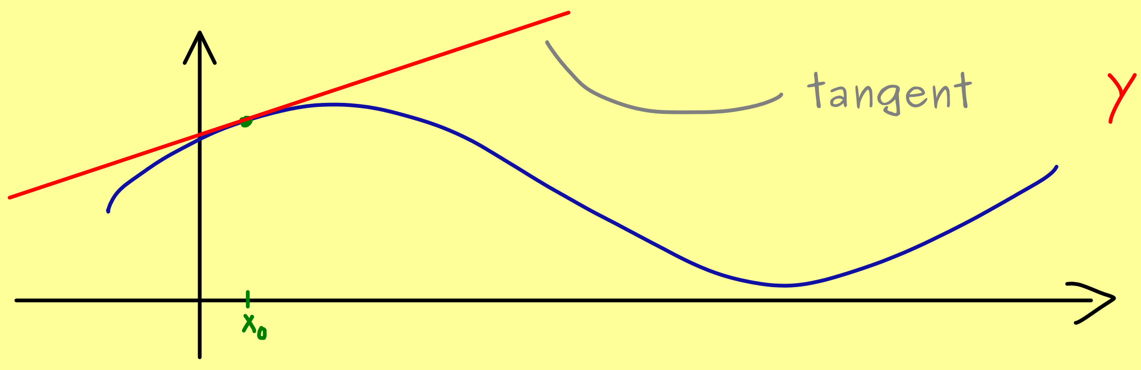

$$\Rightarrow \quad m = \frac{g(x) - g(x_0)}{x - x_0}, \quad x \neq x_0$$

Linear approximation: $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$




$s: \mathbb{R} \to \mathbb{R}$, $s(t) = m \cdot (t - x_0) + c$

$$s(t) = \frac{f(x) - f(x_0)}{x - x_0}(t - x_0) + f(x_0)$$




$y: \mathbb{R} \to \mathbb{R}$

$$y(t) = \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\text{we want it to exist}}(t - x_0) + f(x_0)$$

slope at $x_0$ : $\quad f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: \frac{df}{dx}(x_0)$ differential quotient/ derivative

Definition: $I \subseteq \mathbb{R}$ interval with more than one point

or $I \subseteq \mathbb{R}$ open set, $f: I \to \mathbb{R}$, $x_0 \in I$.

We call $f$ differentiable at $x_0$ if there is a function $\Delta_{f,x_0}: I \to \mathbb{R}$

with $f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x)$ for all $x \in I$

and $\Delta_{f,x_0}$ is continuous at $x_0$.

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## Real Analysis – Part 35

$f$ differentiable at $x_0$ $\iff$ $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists (call it $f'(x_0)$)

$\iff$ $\Delta_{f,x_0}(x) := \frac{f(x)-f(x_0)}{x-x_0}$ for $x \neq x_0$

can be extended to a function that is continuous at $x_0$

$\Delta_{f,x_0}: I \longrightarrow \mathbb{R}$ with $\lim\limits_{x \to x_0} \Delta_{f,x_0}(x) = \Delta_{f,x_0}(x_0)$

$\iff$ There is $\Delta_{f,x_0}: I \longrightarrow \mathbb{R}$ with

$$f(x) = f(x_0) + (x-x_0) \cdot \Delta_{f,x_0}(x) \quad \text{for all } x \in I$$

and $\Delta_{f,x_0}$ is continuous at $x_0$.

$\boxed{\Delta_{f,x_0}(x) = f'(x_0) + r(x)}$

$\iff$ There is $r: I \longrightarrow \mathbb{R}$ and number $b \in \mathbb{R}$ with

$$f(x) = f(x_0) + (x-x_0) \cdot \overset{f'(x_0)}{b} + (x-x_0) \cdot r(x) \quad \text{for all } x \in I$$

and $r$ is continuous at $x_0$ with $r(x_0) = 0$

**Proposition:** $f$ differentiable at $x_0$ $\implies$ $f$ continuous at $x_0$

**Proof:** There is $\Delta_{f,x_0}: I \longrightarrow \mathbb{R}$ which is continuous at $x_0$.

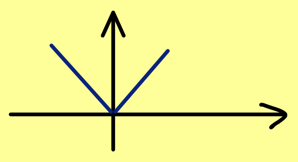
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0}\Big( f(x_0) + (x-x_0) \cdot \Delta_{f,x_0}(x) \Big)$$

$$= f(x_0) + \lim_{x \to x_0} (x-x_0) \cdot \lim_{x \to x_0} \Delta_{f,x_0}(x) = f(x_0) \qquad \square$$

**Examples:** (a) linear polynomial: $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = a_1 \cdot x + a_0$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{a_1 \cdot x + a_0 - (a_1 \cdot x_0 + a_0)}{x - x_0} = \lim_{x \to x_0} \frac{a_1 \cdot \cancel{(x-x_0)}}{\cancel{x-x_0}} = a_1$$

(b) absolute value $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x|$, $x_0 = 0$



$$\left.\begin{aligned} \lim_{x \searrow 0} \frac{f(x)-f(0)}{x-0} &= \lim_{x \searrow 0} \frac{x}{x} = 1 \\ &\qquad \neq \\ \lim_{x \nearrow 0} \frac{f(x)-f(0)}{x-0} &= \lim_{x \nearrow 0} \frac{-x}{x} = -1 \end{aligned}\right\} \implies f \text{ is not differentiable at } 0$$

**Proposition:** $f: I \longrightarrow \mathbb{R}$, $g: I \longrightarrow \mathbb{R}$ differentiable at $x_0$. Then:

(a) $f+g: I \longrightarrow \mathbb{R}$ differentiable at $x_0$ with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

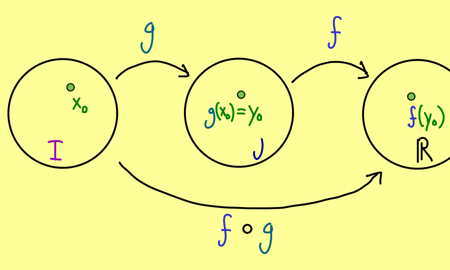
(b) $f \cdot g: I \longrightarrow \mathbb{R}$ differentiable at $x_0$ with $(f \cdot g)'(x_0) = f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)$

**Proof for (b):** $(f \cdot g)(x) = f(x) \cdot g(x) = \big(f(x_0) + (x-x_0)\Delta_{f,x_0}(x)\big) \cdot \big(g(x_0) + (x-x_0)\Delta_{g,x_0}(x)\big)$

$$= f(x_0) \cdot g(x_0) + (x-x_0) \cdot \Big( \underbrace{f(x_0) \Delta_{g,x_0}(x) + \Delta_{f,x_0}(x)\, g(x_0) + (x-x_0) \Delta_{f,x_0}(x) \Delta_{g,x_0}(x)}_{\Delta_{f \cdot g, x_0}(x) \text{ continuous}} \Big)$$

$$(f \cdot g)'(x_0) = f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)$$

## Real Analysis – Part 36




$$(f \circ g)(x) = f(g(x))$$

Chain rule: Let $I, J \subseteq \mathbb{R}$ be two intervals and $g: I \longrightarrow J$, $f: J \longrightarrow \mathbb{R}$.

$$\left.\begin{array}{l} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } y_0 = g(x_0) \end{array}\right\} \Longrightarrow f \circ g \text{ differentiable at } x_0 \text{ and:}$$

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

$$\left.\frac{d f(g(x))}{dx}\right|_{x_0} = \left.\frac{d f(y)}{dy}\right|_{g(x_0)} \cdot \left.\frac{d g(x)}{dx}\right|_{x_0}$$

Proof: $g(x) = g(x_0) + (x - x_0) \cdot \Delta_{g,x_0}(x)$, $\quad f(y) = f(y_0) + (y - y_0) \cdot \Delta_{f,y_0}(y)$, $y_0 = g(x_0)$

$$\begin{aligned} (f \circ g)(x) = f(\underbrace{g(x)}_{y \in J}) &= f(y_0) + (g(x) - y_0) \cdot \Delta_{f,y_0}(g(x)) \\ &= f(y_0) + \big(g(x_0) + (x - x_0) \cdot \Delta_{g,x_0}(x) - y_0\big) \cdot \Delta_{f,y_0}(g(x)) \\ &= f(y_0) + (x - x_0) \cdot \underbrace{\Delta_{g,x_0}(x) \cdot \Delta_{f,y_0}(g(x))}_{\text{continuous at } x_0} \\ &= (f \circ g)(x_0) + (x - x_0) \cdot \Delta_{f \circ g, x_0}(x) \end{aligned}$$

$\Longrightarrow$ $f \circ g$ differentiable at $x_0$ with $(f \circ g)'(x_0) = g'(x_0) \cdot f'(g(x_0)) = f'(g(x_0)) \cdot g'(x_0)$

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# Real Analysis – Part 37

sequence of functions:

$(f_1, f_2, f_3, f_4, f_5, \dots)$ $\qquad f_n : I \to \mathbb{R}$ , $f : I \to \mathbb{R}$

$\boxed{\text{Uniform convergence means: } \quad \|f_n - f\|_\infty \xrightarrow{n \to \infty} 0}$

**Fact:** $f_n$ continuous and $\|f_n - f\|_\infty \xrightarrow{n \to \infty} 0 \quad \Rightarrow \quad f$ continuous

**Definition:** $f : I \to \mathbb{R}$ is called differentiable if $f$ is differentiable at all $x_0 \in I$.

In this case, $f' : I \to \mathbb{R}$ defined by $x \mapsto f'(x)$ is called derivative of $f$

**Example:** $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4 \cdot x + 5 \quad \Rightarrow \quad f' : \mathbb{R} \to \mathbb{R}$ , $f'(x) = 4$

**Theorem:** Let $(f_1, f_2, f_3, f_4, f_5, \dots)$ be a sequence of functions $f_n : I \to \mathbb{R}$.

Assume:
- $(f_n)_{n \in \mathbb{N}}$ is pointwisely convergent to a function $f : I \to \mathbb{R}$
- $f_n : I \to \mathbb{R}$ differentiable for all $n \in \mathbb{N}$
- There is $g : I \to \mathbb{R}$ with $\|f_n' - g\|_\infty \xrightarrow{n \to \infty} 0$

Then: $\|f_n - f\|_\infty \xrightarrow{n \to \infty} 0$ and $f$ differentiable with $f' = g$.

**Proof:** Let $x_0 \in I$.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \underbrace{\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right|} + \underbrace{\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right|}_{\xrightarrow{x \to x_0} 0}$$

$$+ \underbrace{\left| f_n'(x_0) - g(x_0) \right|}_{\xrightarrow{n \to \infty} 0}$$

needs
$\|f_n' - g\|_\infty \xrightarrow{n \to \infty} 0$
mean value theorem is helpful
(see later video!)

For any $\varepsilon > 0$:

$$\left| \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \varepsilon$$

# Real Analysis – Part 38

**Examples:** (a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x \Rightarrow f': \mathbb{R} \to \mathbb{R}$, $f'(x) = 1$

(b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 = x \cdot x$

product rule: $f'(x) = x \cdot 1 + 1 \cdot x = 2 \cdot x$

(c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 = x^2 \cdot x$

product rule: $f'(x) = x^2 \cdot 1 + 2 \cdot x \cdot x = 3 \cdot x^2$

(d) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$

$f'(x) = n \cdot x^{n-1}$ (proof by induction + product rule)

(e) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a_m \cdot x^m + a_{m-1} x^{m-1} + \cdots + a_1 \cdot x^1 + a_0$

$f'(x) = a_m \cdot m \cdot x^{m-1} + a_{m-1} (m-1) \cdot x^{m-2} + \cdots + a_1$

(f) power series: $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$, $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$ ?

**General result for power series:** Let $f: (-r, r) \to \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$, be a power series with radius of convergence $r > 0$.

(1) $\sum_{k=0}^{\infty} a_k \cdot x^k$ is uniformly convergent on each interval $[-c, c] \subseteq (-r, r)$

( sequence of functions $g_n: [-c, c] \to \mathbb{R}$, $g_n(x) = \sum_{k=0}^{n} a_k \cdot x^k$ is uniformly convergent )

(2) $\sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$ is uniformly convergent on each interval $[-c, c] \subseteq (-r, r)$

( sequence of functions $g_n': [-c, c] \to \mathbb{R}$, $g_n'(x) = \sum_{k=1}^{n} a_k \cdot k \cdot x^{k-1}$ is uniformly convergent )

(3) $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$

**Proof:** (1) $\underbrace{\| f - g_n \|_\infty}_{\text{supremum norm on } [-c,c]} = \left\| \sum_{k=n+1}^{\infty} a_k \cdot x^k \right\|_\infty = \sup_{x \in [-c,c]} \lim_{N \to \infty} \left| \sum_{k=n+1}^{N} a_k \cdot x^k \right|$

$\overset{\Delta\text{-inequality}}{\leq} \sup_{x \in [-c,c]} \lim_{N \to \infty} \sum_{k=n+1}^{N} |a_k| \cdot \underbrace{|x|^k}_{\leq c^k} \leq \sum_{k=n+1}^{\infty} |a_k| \cdot c^k \overset{(*)}{\leq} \underbrace{B}_{\text{constant}} \cdot \sum_{k=n+1}^{\infty} \underbrace{q^k}_{|q|<1} \xrightarrow{n \to \infty} 0$

(*) By assumption $\sum_{k=0}^{\infty} a_k \cdot \tilde{r}^k$ is convergent for $c < \tilde{r} < r$.

Hence there is $B$ with $B \geq |a_k \cdot \tilde{r}^k| = |a_k| \cdot \tilde{r}^k = |a_k| \cdot c^k \cdot \Big(\underbrace{\frac{\tilde{r}}{c}}_{q^{-1}}\Big)^k$

$\Rightarrow B \cdot q^k \geq |a_k| \cdot c^k$

(2) Same proof as in (1) because the radius of convergence is the same.

(3) Pointwise convergence of functions + uniform convergence of derivatives:

$\overset{\text{part 37}}{\Longrightarrow}$ $f$ differentiable and $f'(x) = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$

**Examples:** (a) $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \exp'(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \cdot k \cdot x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \cdot x^{\overbrace{k-1}^{\text{new } k}}$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x)$

(b) $\sin(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \Rightarrow \sin'(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} (2m+1) \cdot x^{2m}$

$= \sum_{m=0}^{\infty} (-1)^m \cdot \frac{1}{(2m)!} x^{2m} = \cos(x)$

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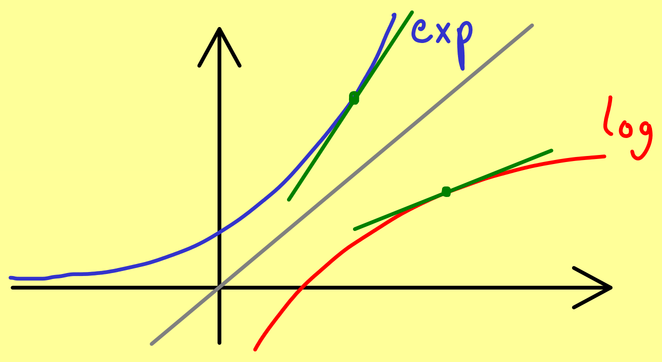
# Real Analysis – Part 39

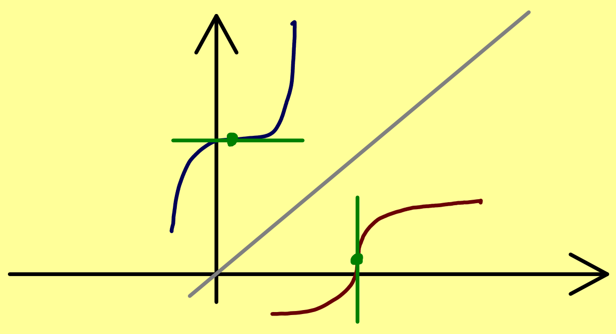
$\log: (0,\infty) \to \mathbb{R}$ defined by the inverse of $\underbrace{\exp: \mathbb{R} \to (0,\infty)}_{\text{differentiable}}$




Consider: $I, J \subseteq \mathbb{R}$ intervals, $f: I \to J$ bijective $\Rightarrow$ $f^{-1}: J \to I$ exists

Assume: $f$ differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$

$y_0 := f(x_0)$



Choose sequence: $(y_n)_{n\in\mathbb{N}} \subseteq J \setminus \{y_0\}$ with $\lim_{n\to\infty} y_n = y_0$

There is exactly one $x_n \in I$ with $f(x_n) = y_n$

$$\frac{f^{-1}(y_n) - f^{-1}(\overset{f(x_0)}{y_0})}{y_n - y_0} = \frac{f^{-1}(f(x_n)) - f^{-1}(f(x_0))}{f(x_n) - f(x_0)} = \frac{x_n - x_0}{f(x_n) - f(x_0)}$$

$$= \left( \frac{f(x_n) - f(x_0)}{x_n - x_0} \right)^{-1}$$

$$(f^{-1})'(y_0) = \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \left( \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \right)^{-1}$$

$$= \left( f'(x_0) \right)^{-1}$$

We need: $x_n \xrightarrow{n\to\infty} x_0$

$\Leftrightarrow$ $f^{-1}(y_n) \xrightarrow{n\to\infty} f^{-1}(y_0)$

$\Leftrightarrow$ $f^{-1}$ continuous at $y_0$

<u>Theorem:</u> Let $I, J \subseteq \mathbb{R}$ be intervals and $f: I \to J$ be bijective.
If $f$ is differentiable at $x_0$ with $f'(x_0) \neq 0$ and $f^{-1}$ is continuous at $y_0 := f(x_0)$,
then $f^{-1}$ is differentiable at $y_0$ with:

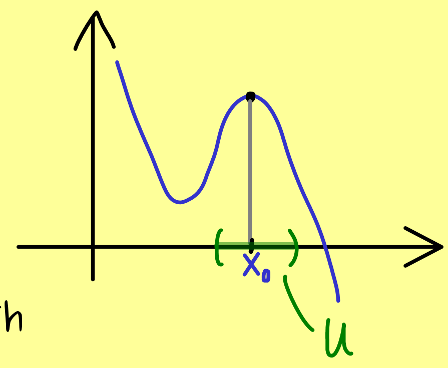
$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

<u>Example:</u> $$\log'(y) = \frac{1}{\exp'(\log(y))} = \frac{1}{\exp(\log(y))} = \frac{1}{y}$$

The Bright Side of Mathematics

# Real Analysis – Part 40

**Definition:** $I \subseteq \mathbb{R}$ interval, $f: I \longrightarrow \mathbb{R}$.



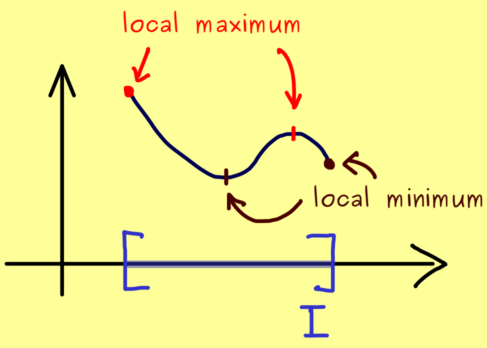

(a) $f$ has a local maximum at $x_0 \in I$ if there is a neighbourhood of $x_0$, $U \subseteq \mathbb{R}$, with

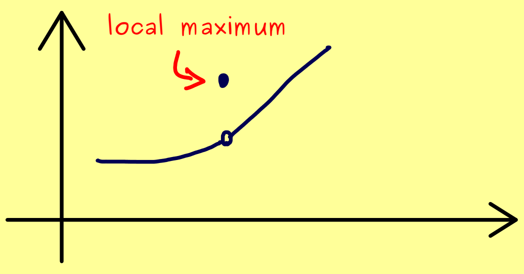
$$f(x_0) = \max\{ f(x) \mid x \in U \cap I \}$$

(b) $f$ has a local minimum at $x_0 \in I$ if there is a neighbourhood of $x_0$, $U \subseteq \mathbb{R}$, with

$$f(x_0) = \min\{ f(x) \mid x \in U \cap I \}$$

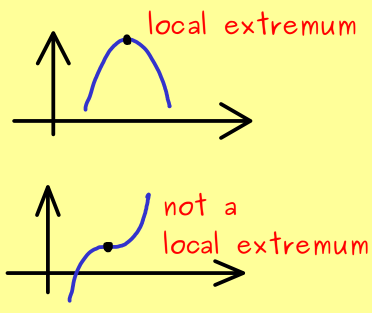
(c) $f$ has a local extremum at $x_0 \in I$ if $f$ has a local maximum or local minimum at $x_0 \in I$.




**Proposition:** $f: (a,b) \longrightarrow \mathbb{R}$ differentiable at $x_0 \in (a,b)$.

$$f \text{ has a local extremum at } x_0 \quad \Rightarrow \quad f'(x_0) = 0$$

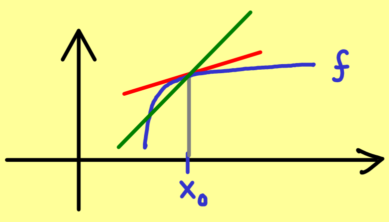



**Proof:** 1st case: $f$ has a local maximum at $x_0$

$\Rightarrow$ there is a neighbourhood of $x_0$, $U \subseteq (a,b)$

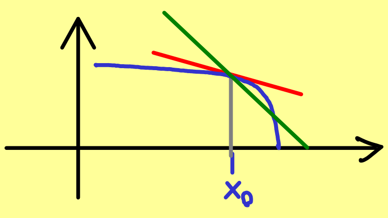
$$f(x_0) = \max\{ f(x) \mid x \in U \}$$

$f$ differentiable at $x_0$ $\Rightarrow$ $f(x) = f(x_0) + (x - x_0) \cdot \underbrace{\Delta_{f,x_0}(x)}_{\text{continuous at } x_0}$




Assume $f'(x_0) > 0$: There exists a neighbourhood $V \subseteq U$ such that $\Delta_{f,x_0}(x) > 0$ for all $x \in V$.

Then: $x > x_0 \Rightarrow f(x) = f(x_0) + \underbrace{(x - x_0)}_{>0} \cdot \underbrace{\Delta_{f,x_0}(x)}_{>0} > f(x_0)$ ↯




Assume $f'(x_0) < 0$: There exists a neighbourhood $V \subseteq U$ such that $\Delta_{f,x_0}(x) < 0$ for all $x \in V$.

Then: $x < x_0 \Rightarrow f(x) = f(x_0) + \underbrace{(x - x_0)}_{<0} \cdot \underbrace{\Delta_{f,x_0}(x)}_{<0} > f(x_0)$ ↯

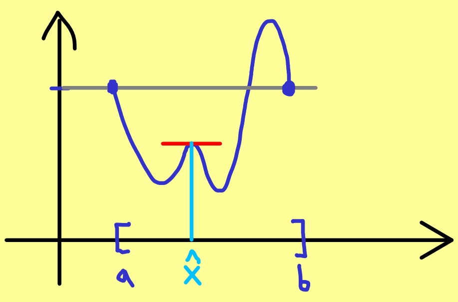
$$\Rightarrow f'(x_0) = 0$$

2nd case: $f$ has a local minimum at $x_0$ (works similarly)

**Theorem of Rolle**

$f: [a,b] \longrightarrow \mathbb{R}$ differentiable and $f(a) = f(b)$.

Then there is $\hat{x} \in (a,b)$ with $f'(\hat{x}) = 0$.




**Proof:** 1st case: $f$ constant $\Rightarrow f'(x) = 0$ for all $x \in [a,b]$. ✓

2nd case: $f$ is not constant.

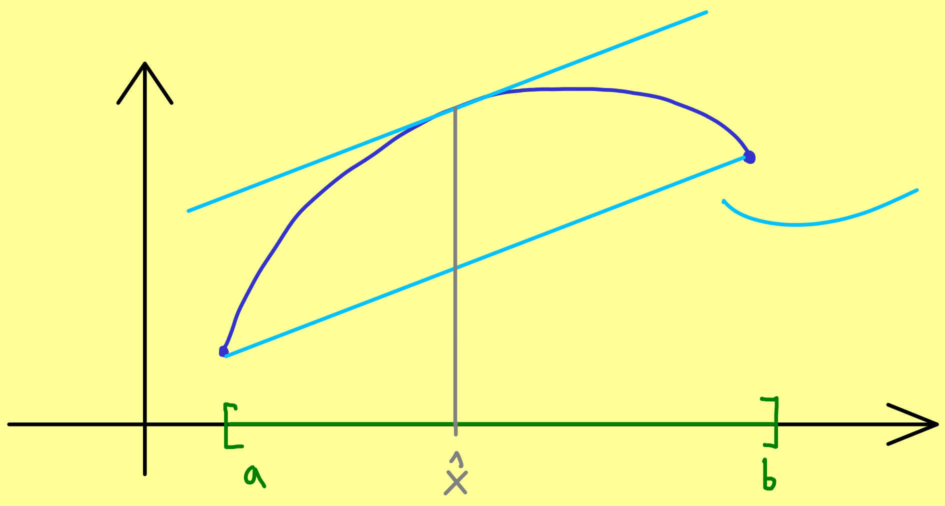
There are $x^-, x^+ \in [a,b]$ with

$$f(x^+) = \sup\{ f(x) \mid x \in [a,b] \}$$
$$f(x^-) = \inf\{ f(x) \mid x \in [a,b] \}$$

$\overset{f \text{ not constant}}{\Longrightarrow}$ $x^- \in (a,b)$ or $x^+ \in (a,b)$ (call it $\hat{x}$)

$\overset{\text{Proposition above}}{\Longrightarrow}$ $f'(\hat{x}) = 0$ $\square$

# Real Analysis – Part 41



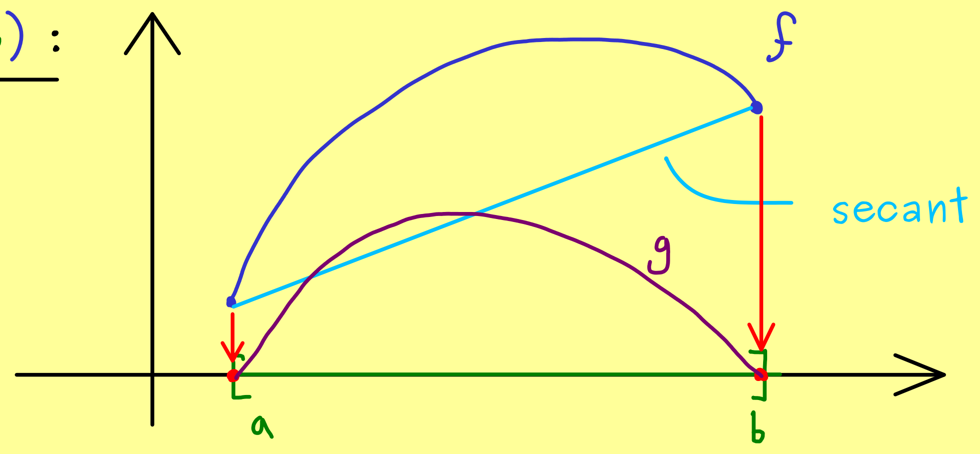

mean slope: $\dfrac{f(b) - f(a)}{b - a}$

**Mean value theorem:** Let $f: [a,b] \to \mathbb{R}$ be differentiable.

Then there exists $\hat{x} \in (a,b)$ with $f'(\hat{x}) = \dfrac{f(b) - f(a)}{b - a}$ .

**Proof:** Rolle's theorem: $f(a) = f(b) \implies$ there is $\hat{x} \in (a,b)$ with $f'(\hat{x}) = \underbrace{\dfrac{f(b) - f(a)}{b - a}}_{= 0}$ ✓

If $f(a) \neq f(b)$ :




Define: $g : [a,b] \to \mathbb{R}$ by $g(x) := f(x) - \left( \dfrac{f(b) - f(a)}{b - a} (x - a) + f(a) \right)$

$\implies g$ differentiable with $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$

Now: $g(a) = g(b) \overset{\text{Rolle's theorem}}{\implies}$ there is $\hat{x} \in (a,b)$ with $g'(\hat{x}) = 0$

$$\implies f'(\hat{x}) = \frac{f(b) - f(a)}{b - a}$$ □

**Application:** $f: [a,b] \to \mathbb{R}$ be differentiable. Assume $f'(x) > 0$ for all $x \in [a,b]$

Then: $x_1 < x_2 \underset{f: [x_1, x_2] \to \mathbb{R}}{\overset{\text{mean value theorem}}{\implies}}$ there is $\hat{x} \in (x_1, x_2)$ with $f'(\hat{x}) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$\implies f(x_2) - f(x_1) = \underbrace{f'(\hat{x})}_{>0} \cdot \underbrace{(x_2 - x_1)}_{>0} > 0$$

$\implies f$ strictly monotonically increasing

(a) $f'(x) > 0$ for all $x \in [a,b] \implies f$ strictly monotonically increasing

(b) $f'(x) < 0$ for all $x \in [a,b] \implies f$ strictly monotonically decreasing

(c) $f'(x) \geq 0$ for all $x \in [a,b] \implies f$ monotonically increasing

(d) $f'(x) \leq 0$ for all $x \in [a,b] \implies f$ monotonically decreasing

# Real Analysis – Part 42

Extended mean value theorem: $f, g : [a,b] \to \mathbb{R}$ be differentiable and $g'(x) \neq 0$ for all $x \in (a,b)$.

Then there exists $\hat{x} \in (a,b)$ with $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\hat{x})}{g'(\hat{x})}$

(If $g(x) = x$, we get the normal mean value theorem)

Proof: We will use Rolle's theorem again.

Define: $h : [a,b] \to \mathbb{R}$ by $h(x) := f(x) - \left( \dfrac{f(b)-f(a)}{g(b)-g(a)} \cdot (g(x) - g(a)) + f(a) \right)$

We have: $h(a) = h(b)$ and $h$ differentiable

$\overset{\text{Rolle's theorem}}{\Longrightarrow}$ there is $\hat{x} \in (a,b)$ with $h'(\hat{x}) = 0$

$$\underbrace{f'(\hat{x}) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(\hat{x})}_{\text{''}} \qquad \square$$

L'Hospital's rule: Let $I$ be an interval and $f, g : I \to \mathbb{R}$ be differentiable.

Let $x_0 \in I$ with $f(x_0) = g(x_0) = 0$ and $g'(x) \neq 0$ for $x \neq x_0$. (at least in a neighbourhood of $x_0$)

Then:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \quad \Longrightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists}$$

$$\text{and} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Proof: Choose sequence $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with $x_n \xrightarrow{n \to \infty} x_0$.

Apply extended mean value theorem for $[a,b] = [x_n, x_0]$ or $= [x_0, x_n]$

$\Longrightarrow$ there is a sequence $(\hat{x}_n)_{n \in \mathbb{N}}$ with $\hat{x}_n \in (x_n, x_0)$ or $(x_0, x_n)$

and $\hat{x}_n \xrightarrow{n \to \infty} x_0$ satisfying:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \xleftarrow{n \to \infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(\hat{x}_n)}{g'(\hat{x}_n)} \xrightarrow{n \to \infty} \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \qquad \square$$

Example:

(a) $$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

(b) $$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{+\sin(x)}{2 \cdot x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

The Bright Side of Mathematics

# Real Analysis – Part 43

## Generalisations of l'Hospital's rule

case "$\frac{0}{0}$"

(a) $I$ interval, $f, g : I \to \mathbb{R}$ differentiable, $x_0 \in I$,
$f(x_0) = g(x_0) = 0$, $g'(x) \neq 0$ for $x \neq x_0$. Then:

$$\boxed{\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

case "$\frac{\infty}{\infty}$"

(b) $I$ interval, $x_0 \in I$, $f, g : I \setminus \{x_0\} \to \mathbb{R}$ differentiable,
$\lim_{x \to x_0} f(x) = \infty$, $\lim_{x \to x_0} g(x) = \infty$. Then:

$$\boxed{\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

case "$\frac{0}{0}$" $x \to \infty$

(c) $I$ interval (with no upper bound), $f, g : I \to \mathbb{R}$ differentiable,
$\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to \infty} g(x) = 0$. Then:

$$\boxed{\lim_{x \to \infty} \frac{f'(x)}{g'(x)} \text{ exists} \implies \lim_{x \to \infty} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}}$$

case "$\frac{\infty}{\infty}$" $x \to \infty$

(d) $I$ interval (with no upper bound), $f, g : I \to \mathbb{R}$ differentiable,
$\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} g(x) = \infty$. Then:

$$\boxed{\lim_{x \to \infty} \frac{f'(x)}{g'(x)} \text{ exists} \implies \lim_{x \to \infty} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}}$$

**Proof:** (b) Use: $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ $\left( \frac{d}{dx} x^{-1} = (-1) \cdot x^{-2} \right)$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

Define: $\tilde{f}(x) := \begin{cases} \frac{1}{f(x)}, & \text{for } x \in I \setminus \{x_0\} \\ 0, & \text{for } x = x_0 \end{cases}$

$\tilde{g}(x) := \begin{cases} \frac{1}{g(x)}, & \text{for } x \in I \setminus \{x_0\} \\ 0, & \text{for } x = x_0 \end{cases}$

(redo proof of l'Hospital's theorem)

$$\frac{\frac{1}{f(x_n)}}{\frac{1}{g(x_n)}} = \frac{\tilde{f}(x_n) - \tilde{f}(x_0)}{\tilde{g}(x_n) - \tilde{g}(x_0)} = \frac{\tilde{f}'(\hat{x}_n)}{\tilde{g}'(\hat{x}_n)} = \frac{\frac{f'(\hat{x}_n)}{(f(\hat{x}_n))^2}}{\frac{g'(\hat{x}_n)}{(g(\hat{x}_n))^2}}$$

(c)

Define: $\tilde{f}(x) := \begin{cases} f\left(\frac{1}{x}\right), & \text{for } x > 0, \; x^{-1} \in I \\ 0, & \text{for } x = 0 \end{cases}$

**Examples:** (1)

$$\lim_{x \to 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) = \lim_{x \to 0} \left( \frac{x - \sin(x)}{x \cdot \sin(x)} \right) \overset{\text{case (a)}}{=} \lim_{x \to 0} \left( \frac{1 - \cos(x)}{\sin(x) + x \cdot \cos(x)} \right)$$

$$= \lim_{x \to 0} \left( \frac{\sin(x)}{\cos(x) + \cos(x) - x \cdot \sin(x)} \right) = 0$$

(2)

$$\lim_{x \to \infty} \left( \frac{x}{\exp(x)} \right) \overset{\text{case (d)}}{=} \lim_{x \to \infty} \left( \frac{1}{\exp(x)} \right) = 0$$

The Bright Side of
Mathematics

## Real Analysis – Part 44

$f: I \to \mathbb{R}$ differentiable $\rightsquigarrow$ $f': I \to \mathbb{R}$

- If $f': I \to \mathbb{R}$ is continuous $\rightsquigarrow$ $f$ continuously differentiable
- If $f': I \to \mathbb{R}$ is differentiable $\rightsquigarrow$ $f$ two-times differentiable

$$f^{(2)} := f'' := (f')' : I \to \mathbb{R}$$

**Definition:** $f: I \to \mathbb{R}$ and set $f^{(0)} := f$. For $n \in \mathbb{N}$, define $f^{(n)} := \left(f^{(n-1)}\right)'$ (inductively)

- $f$ is called n-times differentiable if $f^{(n)}$ exists.
- $f$ is called n-times continuously differentiable if $f^{(n)}$ exists and is continuous.

(Other notations: $f^{(n)} = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f$ )

- $f$ is called $\infty$-times differentiable if $f^{(n)}$ exists for all $n \in \mathbb{N}$.
  (arbitrarily often differentiable)

$$C(I) := \{ f: I \to \mathbb{R} \mid f \text{ continuous} \}$$

$$C^n(I) := \{ f: I \to \mathbb{R} \mid f \text{ n-times continuously differentiable} \}, \quad n \in \mathbb{N} \cup \{\infty\}$$

$$C(I) \supseteq C^1(I) \supseteq C^2(I) \supseteq C^3(I) \supseteq \cdots \supseteq C^\infty(I)$$

Example: $I = \mathbb{R}$
$f(x) = x^2$
exp

**Proposition:** $f: [a,b] \to \mathbb{R}$ differentiable, $x_0 \in [a,b]$, $f'(x_0) = 0$, and $f'$ differentiable at $x_0$. Then:

(a) $f''(x_0) > 0 \implies f$ has a local minimum at $x_0$

(b) $f''(x_0) < 0 \implies f$ has a local maximum at $x_0$

**Proof:** (a) Assume $0 < f''(x_0) = \lim\limits_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \Delta_{f', x_0}(x)$ (continuous at $x_0$)

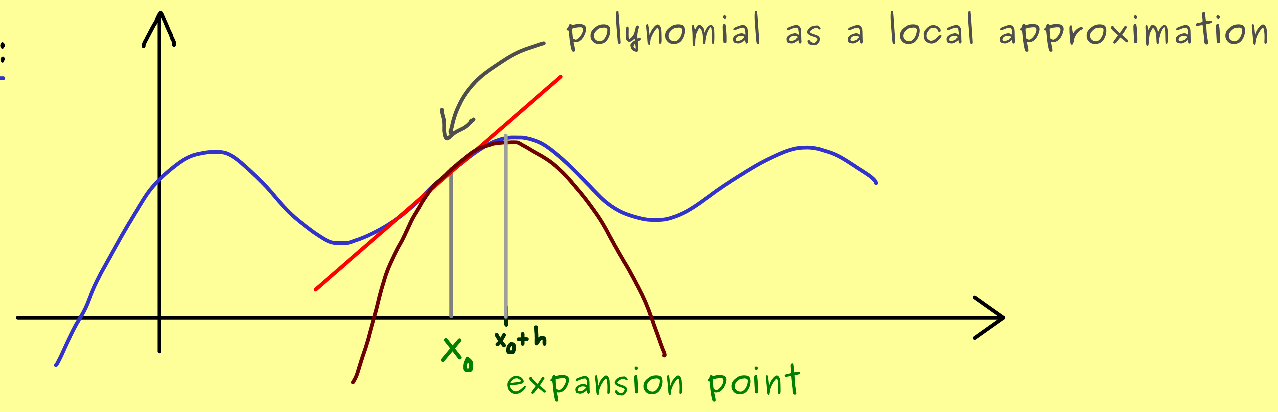
$\implies$ There is a neighbourhood of $x_0$, called $U \subseteq [a,b]$, with $\Delta_{f',x_0}(x) > 0$

$$\implies \left\{ 0 < \frac{f'(x)}{x - x_0} \text{ for } x \in U \setminus \{x_0\} \right. \quad \begin{cases} x < x_0 \Rightarrow f'(x) < 0 \Rightarrow f \text{ decreasing} \\ x > x_0 \Rightarrow f'(x) > 0 \Rightarrow f \text{ increasing} \end{cases}$$

The Bright Side of Mathematics

## Real Analysis – Part 45

**Taylor's theorem:**




Linear approximation: $f(x_0+h) = f(x_0) + f'(x_0)\cdot h + r(h)\cdot h$ with $r(h) \xrightarrow{h\to 0} 0$

$(x = x_0 + h)$

Quadratic approximation: $f(x_0+h) = f(x_0) + f'(x_0)\cdot h + \frac{1}{2}\cdot f''(x_0)\cdot h^2 + r(h)\cdot h^2$

with $r(h) \xrightarrow{h\to 0} 0$

**Theorem:** $I$ interval, $f: I \longrightarrow \mathbb{R}$ $(n+1)$-differentiable, $x_0 \in I$.

If $h \in \mathbb{R}$ such that $x_0 + h \in I$, then:

$$f(x_0+h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k}_{n\text{-th order Taylor polynomial}} + \underbrace{R_n(h)}_{\text{remainder term}}$$

and there is $\xi$ with $\xi \in (x_0, x_0+h)$ or $\xi \in (x_0+h, x_0)$

such that $R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot h^{n+1}$

One often writes: $f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k + \mathcal{O}(h^{n+1})$ (Landau symbol)

Or with $(x = x_0 + h)$: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot (x-x_0)^k + \mathcal{O}((x-x_0)^{n+1})$

The Bright Side of
Mathematics

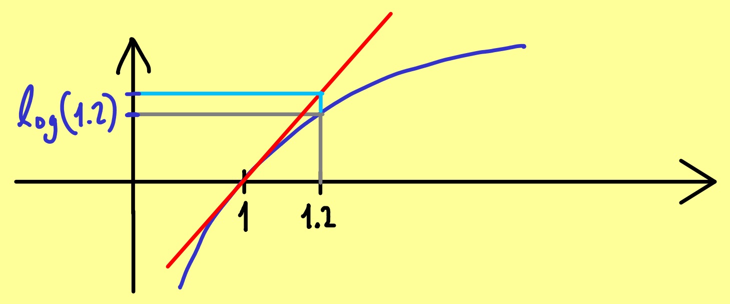
# Real Analysis – Part 46

Taylor: $f(x_0+h) = \underbrace{T_n(h)}_{\displaystyle \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k} + \underbrace{R_n(h)}_{\displaystyle \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot h^{n+1}}$

$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k$: $n$-th order Taylor polynomial

$\xi$ between $x_0$ and $x_0+h$

<u>Example:</u> $\log(1.2) = ?$




expansion point $x_0 = 1$

$h = 0.2$

$\log(x)$   $\log'(x) = \frac{1}{x}$   $\log''(x) = -\frac{1}{x^2}$   $\log'''(x) = \frac{2}{x^3}$   $\log^{(4)}(x) = -\frac{3!}{x^4}$

$\log(x_0) = 0$   $\log'(x_0) = 1$   $\log''(x_0) = -1$   $\log'''(x_0) = 2$

Third order Taylor polynomial: $T_3(h) = 0\cdot h^0 + \frac{1}{1!}h^1 + \frac{-1}{2!}h^2 + \frac{2}{3!}h^3$

$$= h - \frac{1}{2}h^2 + \frac{1}{3}h^3$$

$$T_3(0.2) = \frac{1}{5} - \frac{1}{2}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 = \frac{137}{750} = 0.182\overline{6}$$

first digits of $\log(1.2)$?

$$\left|\log(1.2) - T_3(0.2)\right| = \left|R_3(0.2)\right| = \left|\frac{f^{(3+1)}(\xi)}{(3+1)!}\cdot 0.2^{3+1}\right|$$

$\xi \in (1, 1.2)$

$$= \left|-\frac{3!}{\xi^4}\cdot\frac{1}{4!}\cdot 0.2^4\right| = 4\cdot 10^{-4}\cdot\frac{1}{\xi^4} \leq 0.0004$$

$$0.182\overline{6} - 0.0004 \leq \log(1.2) \leq 0.182\overline{6} + 0.0004$$

$$0.1822 \leq \log(1.2) \leq 0.1831 \quad \Rightarrow \quad \log(1.2) = 0.18\ldots$$

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## Real Analysis – Part 47

**Taylor:** $f(x_0+h) = T_n(h) + R_n(h)$

$$T_n(h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{n\text{-th order Taylor polynomial}} \qquad R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

$\xi$ between $x_0$ and $x_0+h$

**Proof:** $$F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h+x_0-t)^k$$

Note: $F_{n,h}(x_0) = T_n(h)$

$F_{n,h}(x_0+h) = f(x_0+h)$

$$g_{n,h}(t) := (h+x_0-t)^{n+1}, \quad g'_{n,h}(t) = -(n+1)\cdot(h+x_0-t)^n$$

**Generalised mean value theorem:** $$\frac{F_{n,h}(x_0+h) - F_{n,h}(x_0)}{g_{n,h}(x_0+h) - g_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

$\xi$ between $x_0$ and $x_0+h$

$$f(x_0+h) - T_n(h) = \Big(\overbrace{g_{n,h}(x_0+h)}^{0} - \overbrace{g_{n,h}(x_0)}^{h^{n+1}}\Big) \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)} = \frac{h^{n+1}\cdot F'_{n,h}(\xi)}{(n+1)\cdot(h+x_0-\xi)^n}$$

$$\boxed{\begin{aligned} F'_{n,h}(t) &= \frac{d}{dt} \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h+x_0-t)^k \\ &= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!} \cdot (h+x_0-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!} \cdot (h+x_0-t)^{k-1} \\ &= \frac{f^{(n+1)}(t)}{n!} \cdot (h+x_0-t)^n \end{aligned}}$$

$$= \frac{h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot \cancel{(h+x_0-\xi)^n}}{(n+1)\cdot\cancel{(h+x_0-\xi)^n}}$$

$$= h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

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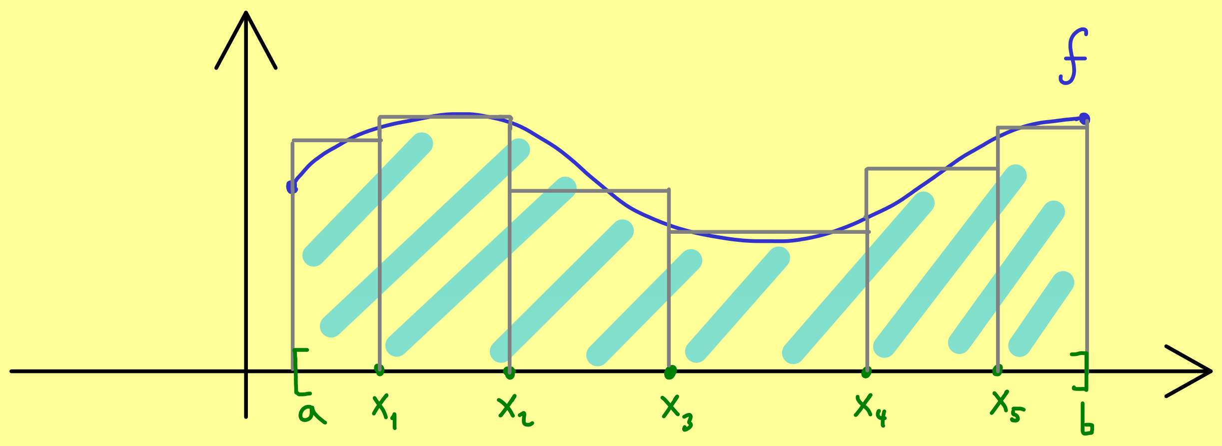
Mathematics

# Real Analysis – Part 48




(orientated)
area between
graph and x-axis

$$\sum_{j=1}^{n} f(\xi_j)\cdot(x_j - x_{j-1})$$

$$\Big\downarrow \; n \to \infty$$

$$\int_a^b f(x)\, dx$$

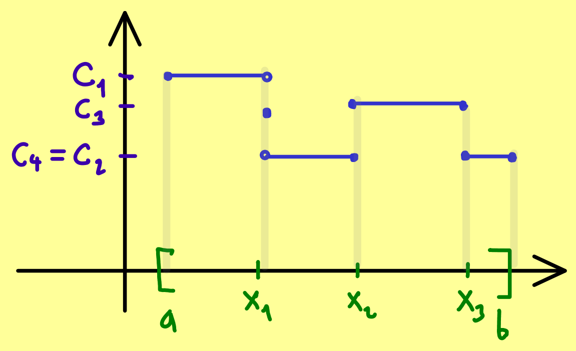
partition of x-axis $\rightsquigarrow$ Riemann integral

(more modern: Lebesgue integral)

**Definition:** partition of $[a,b]$ : a set $\{x_0, x_1, \dots, x_n\}$ with:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

**Definition:** $\phi : [a,b] \longrightarrow \mathbb{R}$ is called a step function if it is piecewisely constant:




there is a partition of $[a,b]$ , $\{x_0, x_1, \dots, x_n\}$ ,
and there are numbers $c_1, \dots, c_n \in \mathbb{R}$ such that

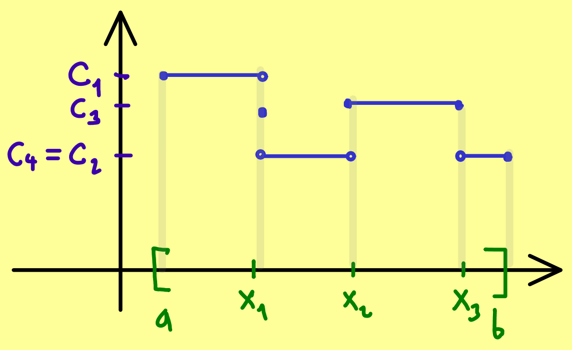
$$\phi|_{(x_{j-1},\, x_j)} = c_j \qquad \text{for all } j \in \{1,\dots,n\}$$

Can we define: $\displaystyle\int_a^b \phi(x)\, dx := \sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1})$ ?

# The Bright Side of Mathematics

## Real Analysis – Part 49

$\phi : [a,b] \longrightarrow \mathbb{R}$ is called a step function if it is piecewisely constant:




there is a partition of $[a,b]$ , $\{x_0, x_1, \dots, x_n\}$ , and there are numbers $c_1, \dots, c_n \in \mathbb{R}$ such that

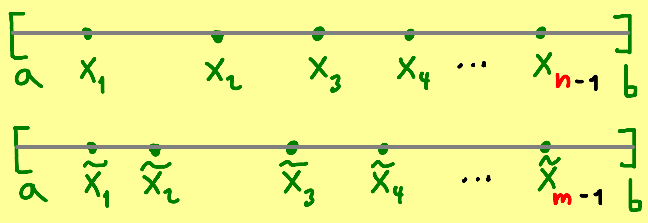
$$\phi\big|_{(x_{j-1}, x_j)} = c_j \qquad \text{for all } j \in \{1, \dots, n\}$$

**Proposition:** $\displaystyle\int_a^b \phi(x)\, dx := \sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1})$ is well-defined.

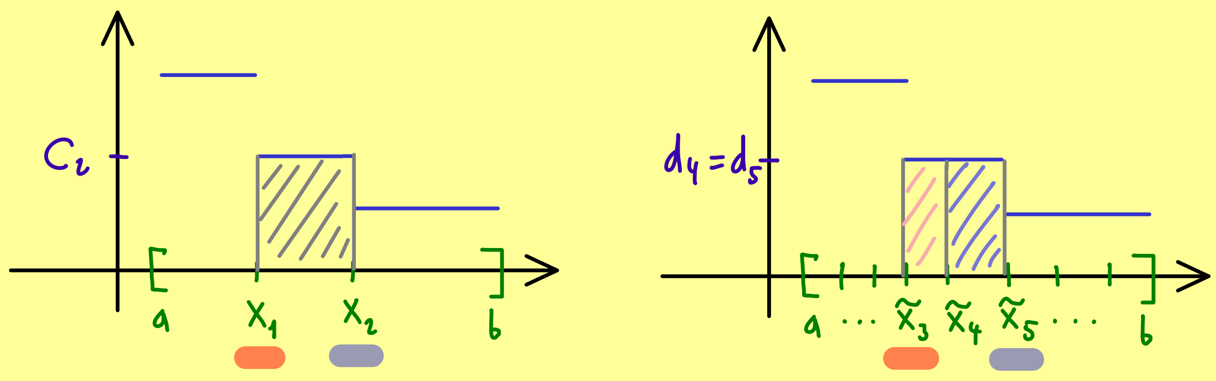
**Proof:**

$P_1 : \quad a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

$P_2 : \quad a = \tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \cdots < \tilde{x}_{m-1} < \tilde{x}_m = b$




with $\phi\big|_{(x_{j-1}, x_j)} = c_j$ , $\phi\big|_{(\tilde{x}_{j-1}, \tilde{x}_j)} = d_j$




**First case:** $P_2 \supset P_1$ (partition 2 is finer than partition 1)

For example: $x_1 = \tilde{x}_3 < \tilde{x}_4 < \tilde{x}_5 = x_2$ , $c_2 = d_4 = d_5$

$$d_4 \cdot (\tilde{x}_4 - \tilde{x}_3) + d_5 \cdot (\tilde{x}_5 - \tilde{x}_4) = c_2 \cdot (\tilde{x}_4 - \underbrace{\tilde{x}_3}_{x_1} + \underbrace{\tilde{x}_5}_{x_2} - \tilde{x}_4) = c_2 \cdot (x_2 - x_1)$$

$$\sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1}) = \sum_{j=1}^{m} d_j \cdot (\tilde{x}_j - \tilde{x}_{j-1})$$

**Second case:** $P_2 \not\supset P_1$ and $P_1 \not\supset P_2$ : $\quad P_3 := P_1 \cup P_2$

$\Rightarrow \quad P_3 \supset P_1$ and $P_3 \supset P_2$

$\Rightarrow \quad \sum_{P_1} = \sum_{P_3}$ and $\sum_{P_2} = \sum_{P_3} \Rightarrow \sum_{P_1} = \sum_{P_2}$

# Real Analysis – Part 50

## Riemann integral for step function: $\int_a^b \phi(x)\,dx$

map: $S([a,b]) \longrightarrow \mathbb{R}$

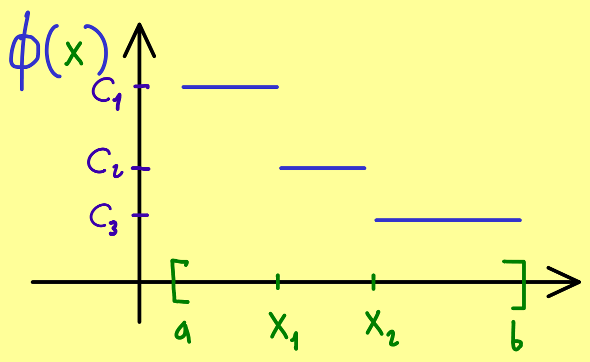
$\phi \longmapsto \int_a^b \phi(x)\,dx$ is linear and monotonic

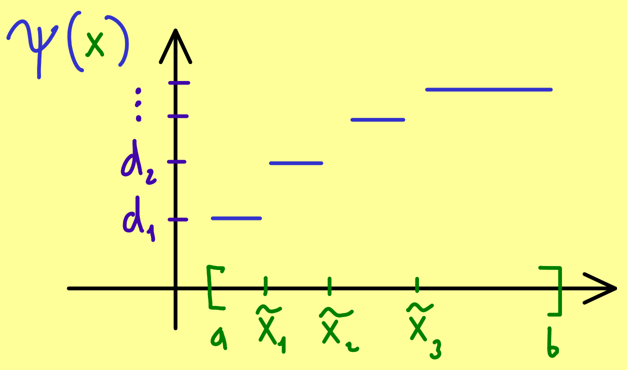
**Proposition:** (1) For $\lambda \in \mathbb{R}$: $\int_a^b \overbrace{\lambda\phi}^{\text{step function}}(x)\,dx = \lambda \cdot \int_a^b \phi(x)\,dx$ (homogeneous)

(2) For $\phi, \psi \in S([a,b])$: $\int_a^b \overbrace{(\phi+\psi)}^{\text{step function}}(x)\,dx = \int_a^b \phi(x)\,dx + \int_a^b \psi(x)\,dx$ (additive)

(3) For $\phi, \psi \in S([a,b])$: $\phi \leq \psi \Rightarrow \int_a^b \phi(x)\,dx \leq \int_a^b \psi(x)\,dx$ (monotonic)

**Proof:** (2)




$P_1: a = x_0 < x_1 < \dots < x_n = b$ $\qquad P_2: a = \tilde{x}_0 < \tilde{x}_1 < \dots < \tilde{x}_m = b$

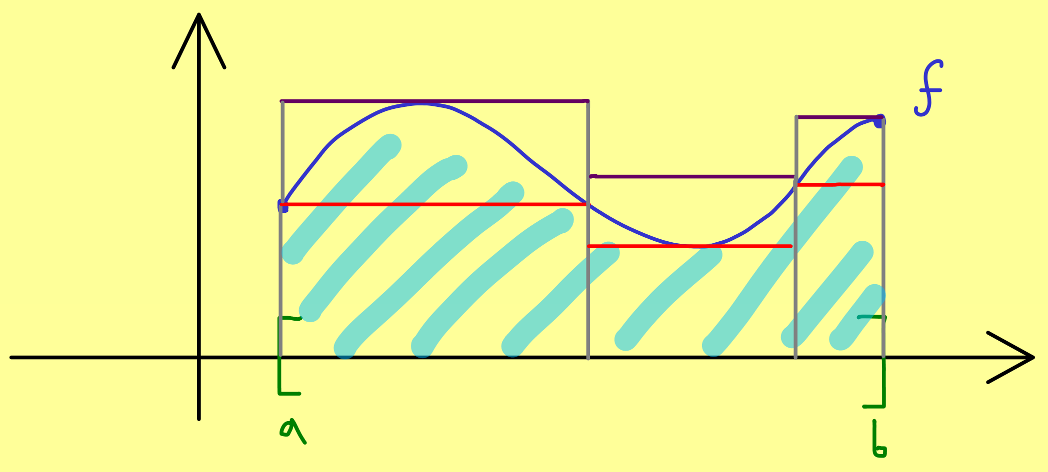
Define: $P_3 = P_1 \cup P_2$ : $a = \tilde{\tilde{x}}_0 < \tilde{\tilde{x}}_1 < \dots < \tilde{\tilde{x}}_N = b$

$$\int_a^b \phi(x)\,dx + \int_a^b \psi(x)\,dx = \sum_{j=1}^{N} c_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) + \sum_{j=1}^{N} d_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1})$$

$$= \sum_{j=1}^{N} (c_j + d_j)(\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) = \int_a^b (\phi+\psi)(x)\,dx$$

The Bright Side of Mathematics

## Real Analysis – Part 51




$f : [a,b] \to \mathbb{R}$ bounded

Use step functions $\phi \in \mathcal{S}([a,b])$ :

$$\sup \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in \mathcal{S}([a,b]) ,\ \phi \leq f \right\}$$

$$\inf \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in \mathcal{S}([a,b]) ,\ \phi \geq f \right\}$$

Definition: A bounded function $f : [a,b] \to \mathbb{R}$ is called Riemann-integrable if

$$\sup \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in \mathcal{S}([a,b]) ,\ \phi \leq f \right\} = \inf \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in \mathcal{S}([a,b]) ,\ \phi \geq f \right\}$$

$$/\!/$$

In this case: $\int_a^b f(x)\,dx$ is called the (Riemann) integral of $f$

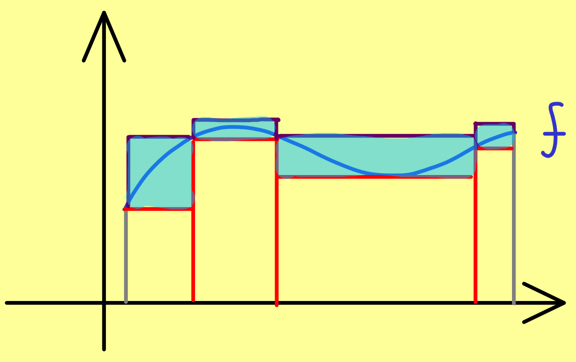
The Bright Side of Mathematics

## Real Analysis – Part 52

**Definition:** A bounded function $f:[a,b]\longrightarrow\mathbb{R}$ is Riemann-integrable

if $\sup\left\{\int_a^b \phi(x)\,dx \;\middle|\; \phi\in S([a,b]),\ \phi\le f\right\}$

$= \inf\left\{\int_a^b \phi(x)\,dx \;\middle|\; \phi\in S([a,b]),\ \phi\ge f\right\}$




$\Longleftrightarrow$ $\forall\varepsilon>0\quad \exists \phi,\psi\in S([a,b]):$

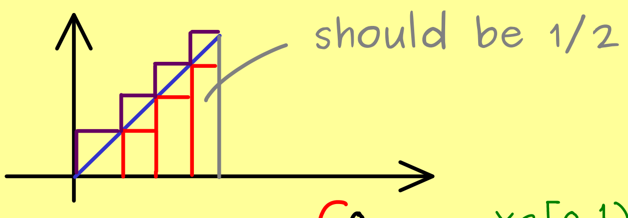
$\phi\le f\le\psi$ and $\int_a^b \psi(x)\,dx-\int_a^b\phi(x)\,dx<\varepsilon$

**Examples:** (a) Dirichlet function $f:[0,1]\longrightarrow\mathbb{R}$, $f(x):=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$


- step function $\psi$ with $f\le\psi$ also satisfies $1\le\psi$
- step function $\phi$ with $\phi\le f$ also satisfies $\phi\le 0$

$\underbrace{\int_a^b\psi(x)\,dx}_{\ge 1}-\underbrace{\int_a^b\phi(x)\,dx}_{\le 0}\ge 1 \quad\Longrightarrow\quad f$ is not Riemann-integrable

(b) $f:[0,1]\longrightarrow\mathbb{R}$, $f(x)=x$




$\phi_4(x)=\begin{cases}0, & x\in[0,\frac14)\\ \frac14, & x\in[\frac14,\frac24)\\ \frac24, & x\in[\frac24,\frac34)\\ \frac34, & x\in[\frac34,\frac44]\end{cases}$

Define $\phi_n(x):=\frac{k-1}{n}$ for $x\in\left[\frac{k-1}{n},\frac{k}{n}\right)$

Then: $\int_0^1\phi_n(x)\,dx=\sum_{k=1}^n\frac{k-1}{n}\cdot\frac1n=\frac{1}{n^2}\sum_{k=1}^n(k-1)=\frac{1}{n^2}\cdot\frac{n\cdot(n-1)}{2}=\frac12-\frac{1}{2n}$

Define $\psi_n(x):=\frac{k}{n}$ for $x\in\left[\frac{k-1}{n},\frac{k}{n}\right)$

Then: $\int_0^1\psi_n(x)\,dx=\sum_{k=1}^n\frac{k}{n}\cdot\frac1n=\frac{1}{n^2}\sum_{k=1}^n k=\frac{1}{n^2}\cdot\frac{n\cdot(n+1)}{2}=\frac12+\frac{1}{2n}$

$\Longrightarrow f$ is Riemann-integrable

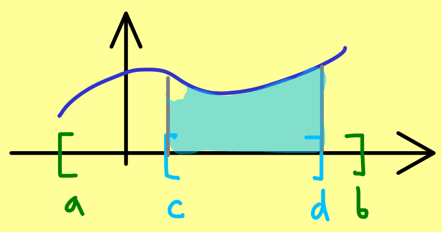
## Real Analysis – Part 53

$$\mathcal{R}([a,b]) := \{ f : [a,b] \to \mathbb{R} \text{ bounded} \mid f \text{ Riemann-integrable} \}$$

**Property (1):** map: $\mathcal{R}([a,b]) \to \mathbb{R}$, $f \mapsto \int_a^b f(x)\,dx$ is linear and monotonic

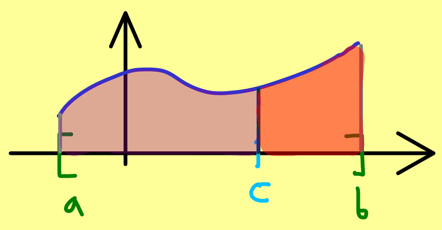
**Definition:** For $c, d \in [a,b]$ with $c < d$,

$$\int_c^d f(x)\,dx := \int_c^d f|_{[c,d]}(x)\,dx$$




**Property (2):** For $c \in [a,b]$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$




**Definition:**

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

**Property (3):** $f \in C([a,b]) \Rightarrow f \in \mathcal{R}([a,b])$

$f$ monotonically increasing $\Rightarrow f \in \mathcal{R}([a,b])$

BECOME A MEMBER

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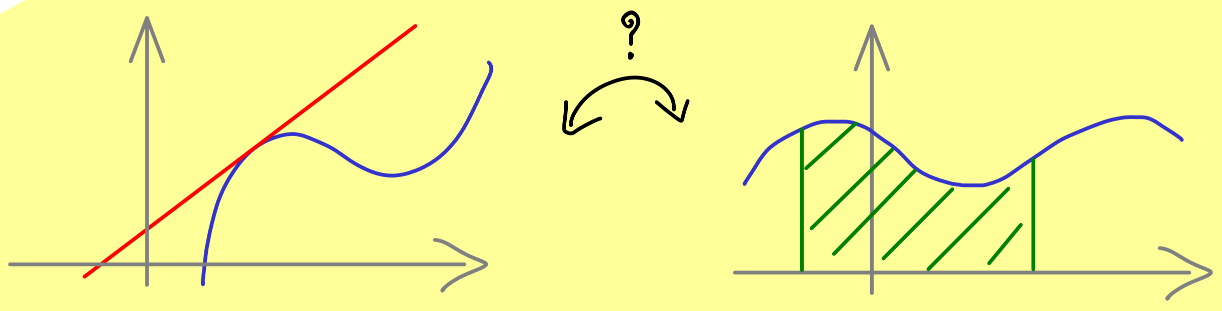
## Real Analysis – Part 54



**Definition:** Let $I \subseteq \mathbb{R}$ be an interval and $f: I \longrightarrow \mathbb{R}$ be a continuous function.

Then a differentiable function $F: I \longrightarrow \mathbb{R}$ is called

an antiderivative of $f$ if

$$F' = f$$

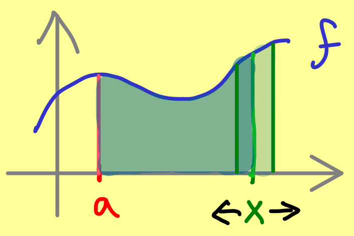
**Theorem:** $I$ interval, $f: I \longrightarrow \mathbb{R}$ continuous, $a \in I$.

first fundamental theorem of calculus

Then $F: I \longrightarrow \mathbb{R}$ defined by $F(x) := \int_a^x f(t)\, dt$

is differentiable and an antiderivative of $f$:

$$F' = f$$

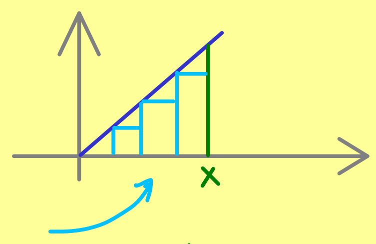



**Examples:** (a) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ $\Rightarrow$ $F(x) = \frac{1}{3}x^3$ is an antiderivative

$F_1(x) = \frac{1}{3}x^3 + 1$ is an antiderivative

for $c \in \mathbb{R}$: $F_c(x) = \frac{1}{3}x^3 + c$ is an antiderivative

(b) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x$, $a = 0$




width: $\frac{x}{n}$, height: $k \cdot \frac{x}{n}$

$$\sum_{k=0}^{n-1} \left(\frac{x}{n}\right) \cdot \left(k \cdot \frac{x}{n}\right) = \frac{x^2}{n^2} \sum_{k=0}^{n-1} k$$

$$= \frac{x^2}{n^2} \frac{(n-1) \cdot n}{2} = \frac{x^2}{2} \cdot \left(1 - \frac{1}{n}\right) \xrightarrow{n \to \infty} \frac{x^2}{2} = \int_0^x f(t)\, dt$$

## Real Analysis – Part 55

**Proposition:** $I$ interval, $f: I \to \mathbb{R}$ continuous, $F: I \to \mathbb{R}$ antiderivative of $f$.

Then: $G: I \to \mathbb{R}$ antiderivative of $f$ $\iff$ $F - G$ is constant

**Proof:** $(\Rightarrow)$ $F, G: I \to \mathbb{R}$ two antiderivatives of $f$

$$(F-G)' = F' - G' = f - f = 0 \overset{\text{mean value theorem}}{\Longrightarrow} F - G \text{ is constant}$$

$(\Leftarrow)$ $F - G$ is constant $\Rightarrow$ $F(x) - G(x) = c$ for a number $c \in \mathbb{R}$

$\Rightarrow G = F - c \Rightarrow G' = F' = f \Rightarrow G$ antiderivative of $f$

**Theorem:** $I$ interval, $f: I \to \mathbb{R}$ continuous, $F: I \to \mathbb{R}$ antiderivative of $f$.

second fundamental theorem of calculus

Then: $$\int_a^b f(t)\,dt = F(b) - F(a) =: F(x)\Big|_a^b$$

**Example:** $$\int_0^1 \cos(x)\,dx = \sin(x)\Big|_0^1 = \sin(1)$$

The Bright Side of Mathematics

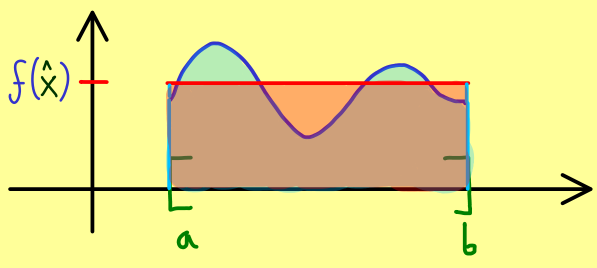
## Real Analysis – Part 56

**Mean value theorem of integration**

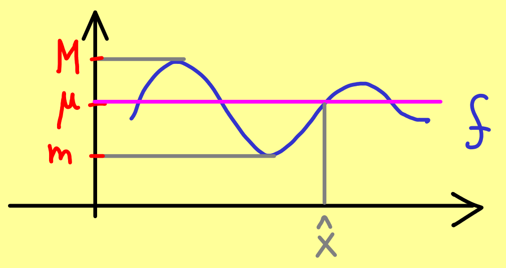
$f, g : [a,b] \to \mathbb{R}$ continuous , $g \geq 0$.

Then there is $\hat{x} \in [a,b]$ with $\int_a^b f(x) g(x)\, dx = f(\hat{x}) \cdot \int_a^b g(x)\, dx$

( often: $g = 1$ : $\int_a^b f(x)\, dx = f(\hat{x}) \cdot (b-a)$ )




**Proof:**




$m \leq f(x) \leq M$ for all $x \in [a,b]$

$g \geq 0 \quad \Rightarrow \quad m\, g(x) \leq f(x) g(x) \leq M\, g(x)$

monotonicity: $m \int_a^b g(x)\, dx \leq \int_a^b f(x) g(x)\, dx \leq M \int_a^b g(x)\, dx$

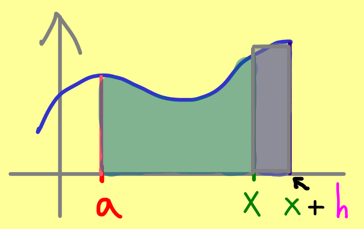
there is $\mu \in [m, M]$ : $\mu \int_a^b g(x)\, dx = \int_a^b f(x) g(x)\, dx$

$\overset{\text{intermediate value theorem}}{\Longrightarrow}$ there is $\hat{x} \in [a,b]$ with $f(\hat{x}) = \mu$ $\square$

**Proof of the first fundamental theorem of calculus:**

$F(x) := \int_a^x f(t)\, dt$

$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt$




$= f(\hat{x}) \cdot h$ with $\hat{x} \in [x, x+h]$ (or $\hat{x} \in [x+h, x]$ )

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(\hat{x}) = f(x) \Rightarrow F' = f$ $\square$

**Proof of the second fundamental theorem of calculus:**

$\boxed{\int_a^b f(t)\, dt = F(b) - F(a)}$

holds for $F_0$ ✓

$F_0(x) := \int_a^x f(t)\, dt$ antiderivative of $f$ with $F_0(a) = 0$

arbitrary antiderivative of $f$: $F = F_0 + c$ for $c \in \mathbb{R}$

$F(b) - F(a) = F_0(b) - F_0(a) = \int_a^b f(t)\, dt$ $\square$

The Bright Side of Mathematics

## Real Analysis – Part 57

### Integration by substitution

$I \subseteq \mathbb{R}$ interval, $f: I \to \mathbb{R}$ continuous, $\phi: [a,b] \to I$ continuously differentiable

Then: $$\int_a^b f(\underbrace{\phi(t)}_{= x}) \cdot \underbrace{\phi'(t)\, dt}_{dx} = \int_{\phi(a)}^{\phi(b)} f(x)\, dx$$

Remember: $x = \phi(t)$

$$\frac{dx}{dt} = \phi'(t) \Rightarrow dx = \phi'(t)\, dt$$

Example: $$\int_0^1 t^2 \cdot \sin(t^3)\, dt = \frac{1}{3} \int_0^1 \sin(t^3) \cdot 3t^2\, dt = \frac{1}{3} \int_0^1 \sin(x)\, dx$$

$$\boxed{\begin{aligned} x &= t^3 \\ dx &= 3t^2\, dt \end{aligned}}$$

Proof: Let $F: I \to \mathbb{R}$ be an antiderivative of $f$

$$(F \circ \phi)'(t) \overset{\text{chain rule}}{=} F'(\phi(t)) \cdot \phi'(t) = f(\phi(t)) \cdot \phi'(t)$$

$$\int_a^b f(\phi(t)) \cdot \phi'(t)\, dt = \int_a^b (F \circ \phi)'(t)\, dt = (F \circ \phi)(t) \Big|_{t=a}^{t=b}$$

$$= F(x) \Big|_{x=\phi(a)}^{x=\phi(b)} = \int_{\phi(a)}^{\phi(b)} f(x)\, dx \qquad \square$$

Another substitution rule: $f: [a,b] \to \mathbb{R}$ continuous, $\phi: J \to I$ continuously differentiable and bijective

$J, I \subseteq \mathbb{R}$ intervals, $I \supseteq [a,b]$

$$\int_a^b f(x)\, dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t)) \cdot \phi'(t)\, dt$$

Example: $\int_0^b \frac{1}{\sqrt{1-x^2}}\, dx$   $b \in [0,1)$

bijective: $[0, \frac{\pi}{2}] \to [0,1]$

try: $\phi(t) = \sin(t)$

substitution: $x = \sin(t)$

$dx = \cos(t)\, dt$

$\sin^2(t) + \cos^2(t) = 1$

$$= \int_0^{\arcsin(b)} \frac{1}{\underbrace{\sqrt{1-(\sin(t))^2}}_{\cos(t)}} \cos(t)\, dt = \int_0^{\arcsin(b)} 1\, dt = \arcsin(b)$$

# Real Analysis – Part 58

## Integration by parts

$I \subseteq \mathbb{R}$ interval, $f, g : I \to \mathbb{R}$ continuously differentiable, $a, b \in I$

Then: $$\int_a^b f'(x) \cdot g(x)\, dx = f(x) \cdot g(x) \Big|_{x=a}^{x=b} - \int_a^b f(x) \cdot g'(x)\, dx$$

**Example:**

$$\int_a^b \overset{g(x)}{\overbrace{x}} \cdot \underbrace{\exp(x)}_{f'(x)}\, dx = x \cdot \exp(x) \Big|_{x=a}^{x=b} - \int_a^b \exp(x) \cdot 1\, dx$$

$$= x \cdot \exp(x) \Big|_{x=a}^{x=b} - \exp(x) \Big|_{x=a}^{x=b}$$

$$= \Big( x \cdot \exp(x) - \exp(x) \Big) \Big|_{x=a}^{x=b}$$

$f'(x) = \exp(x)$
$g(x) = x$
$f(x) = \exp(x)$
$g'(x) = 1$

**Proof:** product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$f(x) \cdot g(x) \Big|_{x=a}^{x=b} \overset{\text{fundamental theorem of calculus}}{=} \int_a^b (f \cdot g)'(x)\, dx = \int_a^b f'(x) \cdot g(x)\, dx + \int_a^b f(x) \cdot g'(x)\, dx \qquad \square$$

The Bright Side of Mathematics

## Real Analysis – Part 59

$$\int_3^5 \frac{1}{x(x+1)}\,dx = ? \qquad \text{antiderivative?}$$

partial fraction decomposition

$$\frac{1}{x(x+1)} \overset{\checkmark}{=} \frac{1}{x} - \frac{1}{x+1} \qquad \left( = \frac{1\cdot(x+1)}{x\cdot(x+1)} - \frac{x\cdot 1}{x(x+1)} \right)$$

antiderivative: $$\int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \log(|x|) - \log(|x+1|) \quad + \text{constant}$$

Partial fraction decomposition: Let $f$ be a rational function

$$f(x) = \frac{p(x)}{q(x)} \quad \text{with} \quad \deg(p) < \deg(q) =: n$$

We need the zeros of $q$:

(1) $n$ different real zeros: $x_1, x_2, \dots, x_n$

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \dots + \frac{A_n}{x-x_n} \qquad \text{Find } A_1, \dots, A_n\,!$$

(2) $k$ different real zeros: $x_1, x_2, \dots, x_k$ with multiplicities $\alpha_1, \dots, \alpha_k$

$$\sum_{j=1}^{k} \alpha_j = n$$

$$\frac{p(x)}{q(x)} = \frac{A_1^{(1)}}{x-x_1} + \frac{A_1^{(2)}}{(x-x_1)^2} + \dots + \frac{A_1^{(\alpha_1)}}{(x-x_1)^{\alpha_1}} + \frac{A_2^{(1)}}{x-x_2} + \frac{A_2^{(2)}}{(x-x_2)^2} + \dots$$

(3) $q$ has complex zeros: calculate as in (1) and (2) with

$$x_1, x_2, \dots, x_k \in \mathbb{C}, \quad A_1^{(1)}, \dots, A_k^{(\alpha_k)} \in \mathbb{C}$$

Example: $f(x) = \dfrac{1}{x^2(x-1)}$, zeros of the denominator: $x_1 = 0$, $x_2 = 1$; $\alpha_1 = 2$, $\alpha_2 = 1$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \qquad \Big|\cdot x^2(x-1)$$

$$\Rightarrow \quad 1 = A\cdot x(x-1) + B\cdot(x-1) + C\cdot x^2$$

$$\Rightarrow \quad 1 = x^2\cdot(A+C) + x\cdot(-A+B) + 1\cdot(-B)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{II+I}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right) \overset{III+II}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \quad C = 1, \quad B = -1, \quad A = -1$$

$$\Rightarrow \quad \int \frac{1}{x^2(x-1)}\,dx = \int \frac{-1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{1}{(x-1)}\,dx$$

$$= -\log(|x|) + \frac{1}{x} + \log(|x-1|) \quad + \text{constant}$$

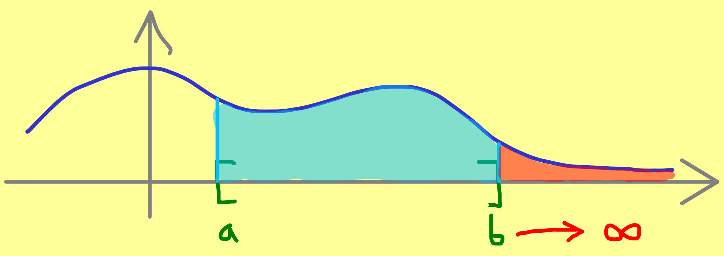
The Bright Side of Mathematics

## Real Analysis – Part 60

$f : \mathbb{R} \to \mathbb{R}$ continuous

$\int_a^b f(x)\, dx$ well-defined for $a, b \in \mathbb{R}$

$\int_a^\infty f(x)\, dx = \, ?$

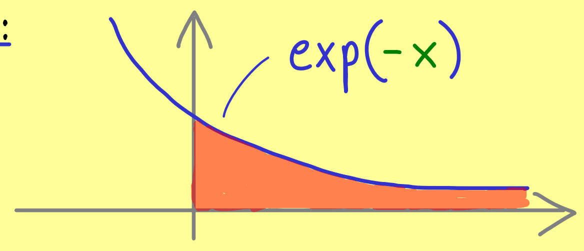



**Definition:** $f : [a, \infty) \to \mathbb{R}$ be a function with the property:

$$f|_{[a,b]} \in \mathcal{R}([a,b]) \quad \text{for all } b \geq a$$

If $\lim_{b \to \infty} \int_a^b f(x)\, dx$ exists, we write $\int_a^\infty f(x)\, dx$ for this limit and we say the integral converges.

**Example:**




$$\int_0^\infty \exp(-x)\, dx = \lim_{b \to \infty} \int_0^b \exp(-x)\, dx$$

$$= \lim_{b \to \infty} \left( -\exp(-x) \Big|_0^b \right)$$

$$= \lim_{b \to \infty} \left( -\exp(-b) + 1 \right) = 1$$

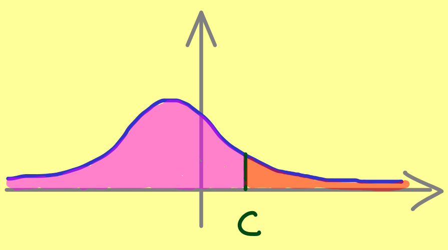
Similar definition for: $\int_{-\infty}^b f(x)\, dx$

**Definition:** $f : \mathbb{R} \to \mathbb{R}$ be a function with the property:

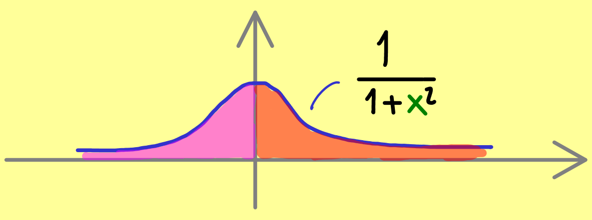
$$f|_{[a,b]} \in \mathcal{R}([a,b]) \quad \text{for all } a, b \in \mathbb{R} \quad (a < b)$$

If there is a $c \in \mathbb{R}$ such that $\int_{-\infty}^c f(x)\, dx$ and $\int_c^\infty f(x)\, dx$ converge,

$$\int_{-\infty}^\infty f(x)\, dx := \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$$




**Example:**




$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = \int_{-\infty}^0 \frac{1}{1+x^2}\, dx + \int_0^\infty \frac{1}{1+x^2}\, dx = \lim_{a \to -\infty} \int_a^0 \frac{1}{1+x^2}\, dx + \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2}\, dx$$

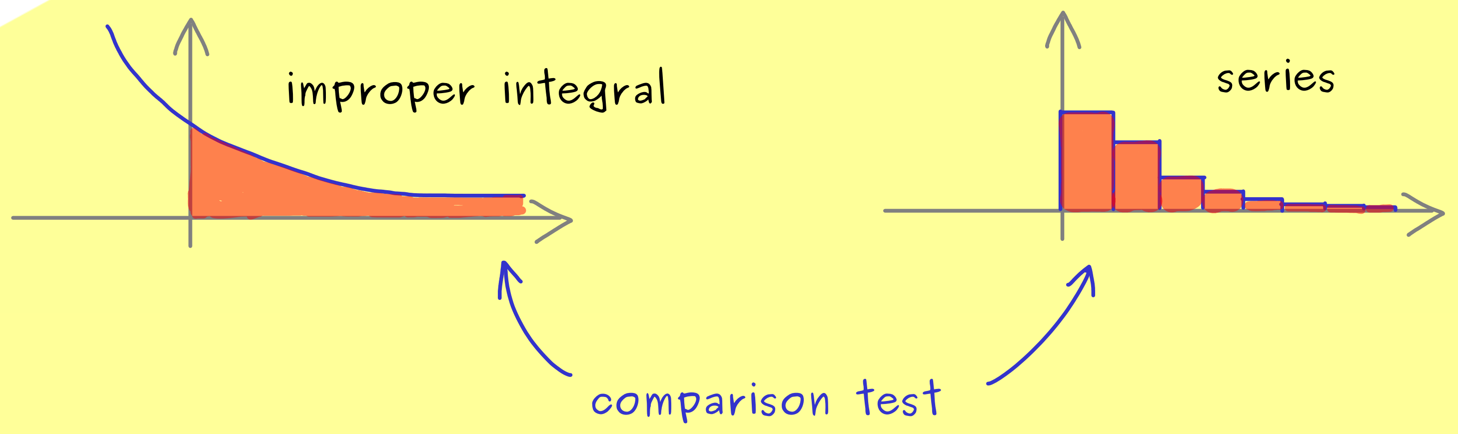
$$= \lim_{a \to -\infty} \arctan(x) \Big|_a^0 + \lim_{b \to \infty} \arctan(x) \Big|_0^b$$

$$= \lim_{b \to \infty} \arctan(b) - \lim_{a \to -\infty} \arctan(a)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

# The Bright Side of Mathematics

## Real Analysis – Part 61




<u>Theorem:</u> $f, g : [a, \infty) \to \mathbb{R}$ with $g(x) \geq 0$ for all $x \in [a, \infty)$ and:

$$g|_{[a,b]}, f|_{[a,b]} \in \mathcal{R}([a,b]) \quad \text{for all } b \geq a .$$

(a) If $|f(x)| \leq g(x)$ for all $x \in [a, \infty)$, then:

$$\int_a^\infty g(x)\,dx \text{ converges} \implies \int_a^\infty f(x)\,dx \text{ converges}$$

(b) If $g(x) \leq f(x)$ for all $x \in [a, \infty)$, then:

$$\int_a^\infty g(x)\,dx \text{ diverges} \implies \int_a^\infty f(x)\,dx \text{ diverges}$$

<u>Example:</u> Recall: $\int_1^\infty \frac{1}{x}\,dx$ diverges since $\int_1^b \frac{1}{x}\,dx = \log(b) \xrightarrow{b \to \infty} \infty$

Is $\int_1^\infty \frac{x}{x^2+1}\,dx$ convergent?

$$x \cdot \left(\frac{x}{x^2+1}\right) = \frac{x^2}{x^2+1} = \frac{1}{1+\frac{1}{x^2}} \xrightarrow{x \to \infty} 1$$

so eventually: $x \cdot \left(\frac{x}{x^2+1}\right) \geq \frac{1}{2}$

there is $R \geq 1$ such that for all $x \geq R$:

$$\frac{x}{x^2+1} \geq \frac{1}{2} \cdot \frac{1}{x}$$

$\int_R^\infty \frac{x}{x^2+1}\,dx$ is divergent because $\int_R^\infty \frac{1}{2} \cdot \frac{1}{x}\,dx$ is divergent

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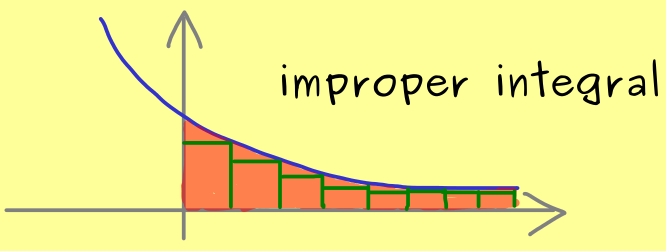
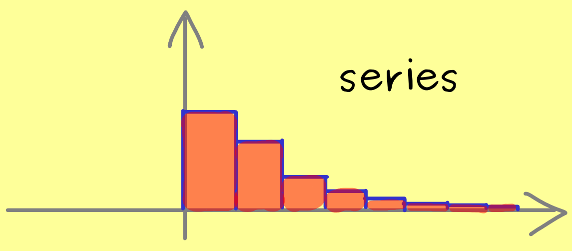
## Real Analysis – Part 62

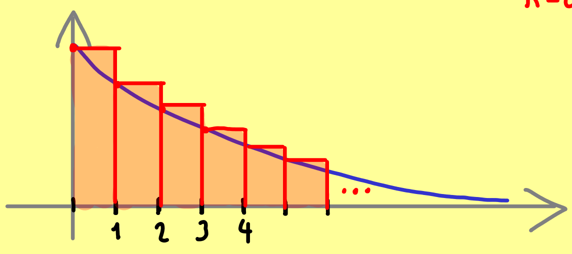


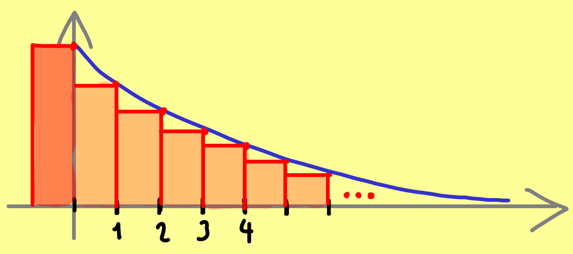




**Theorem:** Let $f: [0,\infty) \longrightarrow [0,\infty)$ be monotonically decreasing.

Then: $\sum_{k=0}^{\infty} f(k)$ convergent $\Longleftrightarrow$ $\int_0^{\infty} f(x)\, dx$ convergent

In this case: $0 \leq \sum_{k=0}^{\infty} f(k) - \int_0^{\infty} f(x)\, dx \leq f(0)$




**Proof:**
$$f(k) = \int_{k-1}^{k} f(k)\, dx \leq \int_{k-1}^{k} f(x)\, dx \leq \int_{k-1}^{k} f(k-1)\, dx = f(k-1)$$

$$\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} \int_{k-1}^{k} f(x)\, dx \leq \sum_{k=1}^{n} f(k-1)$$

$$\Longrightarrow \sum_{k=1}^{n} f(k) \leq \int_0^{n} f(x)\, dx \leq \sum_{k=0}^{n-1} f(k)$$
( $n \to \infty$ shows first part)

If the limits exist: $\sum_{k=1}^{\infty} f(k) \leq \int_0^{\infty} f(x)\, dx \leq \sum_{k=0}^{\infty} f(k)$ $\square$

**Example:**
$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} \begin{cases} \text{convergent for } \alpha > 1 \\ \text{divergent for } 0 < \alpha \leq 1 \end{cases}$$

**Proof:**
$$\int_1^b \frac{1}{x^{\alpha}}\, dx = \begin{cases} \frac{1}{1-\alpha} x^{-\alpha+1} \Big|_1^b, & \alpha \neq 1 \\ \log(x) \Big|_1^b, & \alpha = 1 \end{cases}$$

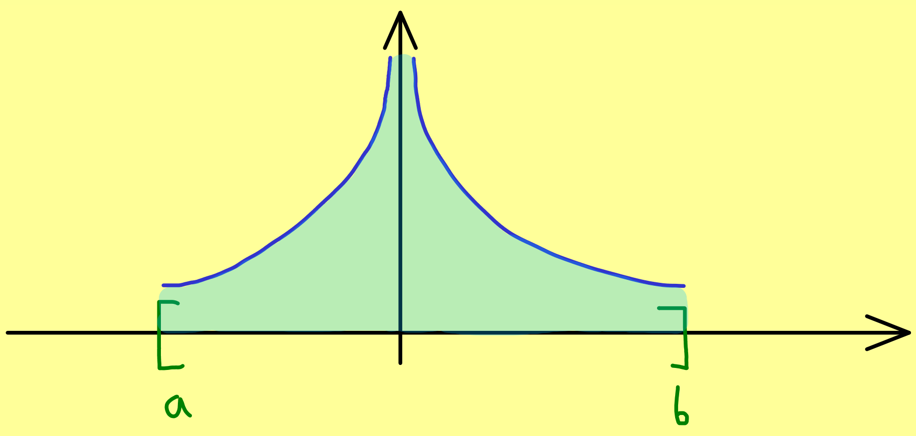
$$= \begin{cases} \frac{1}{1-\alpha} b^{-(\alpha-1)} - \frac{1}{1-\alpha}, & \alpha > 1 \\ \frac{1}{1-\alpha} b^{1-\alpha} - \frac{1}{1-\alpha}, & \alpha < 1 \\ \log(b), & \alpha = 1 \end{cases} \xrightarrow{b \to \infty} \begin{cases} \frac{1}{\alpha - 1}, & \alpha > 1 \\ \infty, & \alpha < 1 \\ \infty, & \alpha = 1 \end{cases}$$

## Real Analysis – Part 63

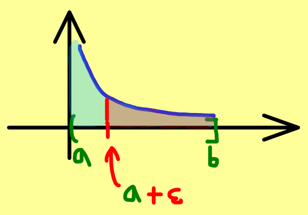
We know: $f : [a,b] \longrightarrow \mathbb{R}$ Riemann-integrable

$\Longrightarrow$ $f$ is bounded

What about this?




**Definition:** Let $f : (a,b] \longrightarrow \mathbb{R}$ be a function with the property that




$$f\big|_{[a+\varepsilon, b]} \in \mathcal{R}\big([a+\varepsilon, b]\big) \quad \text{for all } \varepsilon > 0 .$$

If $\lim\limits_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx$ exists, we write $\int_a^b f(x)\,dx$ for this limit and we say the integral converges.

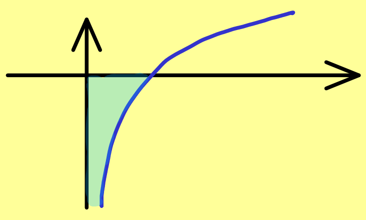
**Example:** $\int_0^1 \log(x)\,dx$ ?

$$\boxed{\begin{aligned} f'(x) &= 1 \\ g(x) &= \log(x) \\ f(x) &= x \\ g'(x) &= \tfrac{1}{x} \end{aligned}}$$

integration by parts

$$\int_\varepsilon^1 \log(x)\,dx = \int_\varepsilon^1 1 \cdot \log(x)\,dx = x \cdot \log(x)\Big|_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x}\,dx = x \cdot \big(\log(x) - 1\big)\Big|_\varepsilon^1$$

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^1 \log(x)\,dx = 1 \cdot \big(\log(1) - 1\big) - \lim_{\varepsilon \searrow 0} \varepsilon \cdot \big(\log(\varepsilon) - 1\big)$$



$$= -1 - \lim_{\varepsilon \searrow 0} \varepsilon \cdot \log(\varepsilon) = -1 - \lim_{\varepsilon \searrow 0} \frac{\log(\varepsilon)}{\frac{1}{\varepsilon}} = -1$$

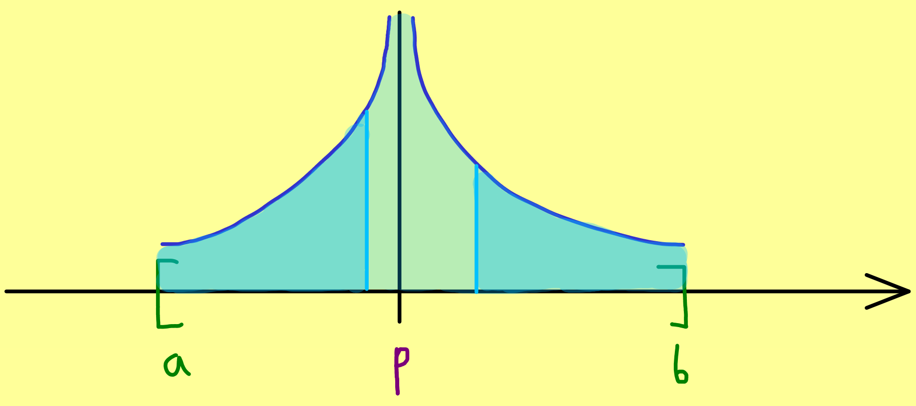
The Bright Side of Mathematics

## Real Analysis – Part 64

For $f : [a,b] \setminus \{p\} \longrightarrow \mathbb{R}$

one defines the following

improper Riemann integral:

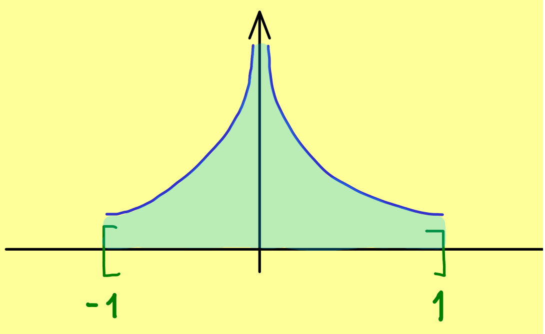



$$\int_a^b f(x)\,dx := \lim_{\varepsilon_1 \searrow 0} \int_a^{p-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{p+\varepsilon_2}^{b} f(x)\,dx$$

Example:

$$\int_{-1}^{1} \frac{1}{2\sqrt{|x|}}\,dx = \lim_{\varepsilon_1 \searrow 0} \int_{-1}^{-\varepsilon_1} \frac{1}{2\sqrt{|x|}}\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{\varepsilon_2}^{1} \frac{1}{2\sqrt{|x|}}\,dx$$

$$= \lim_{\varepsilon_1 \searrow 0} \int_{-1}^{-\varepsilon_1} \frac{1}{2\sqrt{-x}}\,dx + \lim_{\varepsilon_2 \searrow 0} \int_{\varepsilon_2}^{1} \frac{1}{2\sqrt{x}}\,dx$$

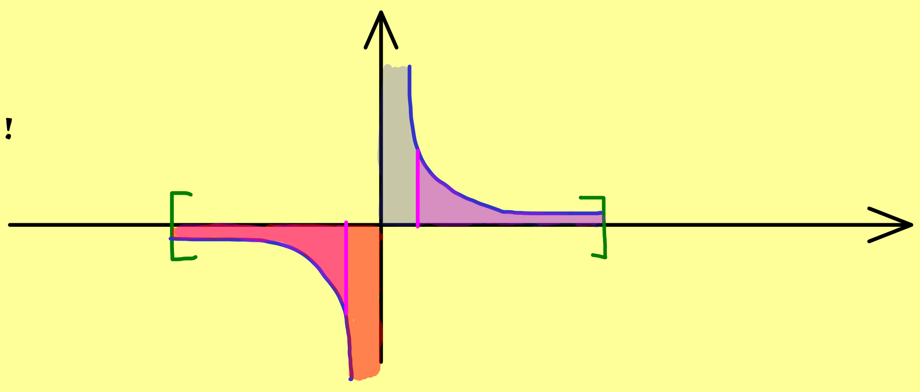



$$= \lim_{\varepsilon_1 \searrow 0} \left( -\sqrt{-x} \,\Big|_{-1}^{-\varepsilon_1} \right) + \lim_{\varepsilon_2 \searrow 0} \left( \sqrt{x} \,\Big|_{\varepsilon_2}^{1} \right)$$

$$= \lim_{\varepsilon_1 \searrow 0} \left( -\sqrt{\varepsilon_1} - \left(-\sqrt{1}\right) \right) + \lim_{\varepsilon_2 \searrow 0} \left( \sqrt{1} - \sqrt{\varepsilon_2} \right) = 2$$

Counterexample:

$$\int_{-1}^{1} \frac{1}{x}\,dx \quad \text{does not exist!}$$



Cauchy principal value:

$$\text{p.v.}\int_{-1}^{1} \frac{1}{x}\,dx := \lim_{\varepsilon \searrow 0} \left( \int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right)$$

$$= \lim_{\varepsilon \searrow 0} \left( \log(|x|)\Big|_{-1}^{-\varepsilon} + \log(|x|)\Big|_{\varepsilon}^{1} \right) = 0$$

$$\text{p.v.}\int_{-\infty}^{\infty} x\,dx = \lim_{a \to \infty} \int_{-a}^{a} x\,dx = 0$$