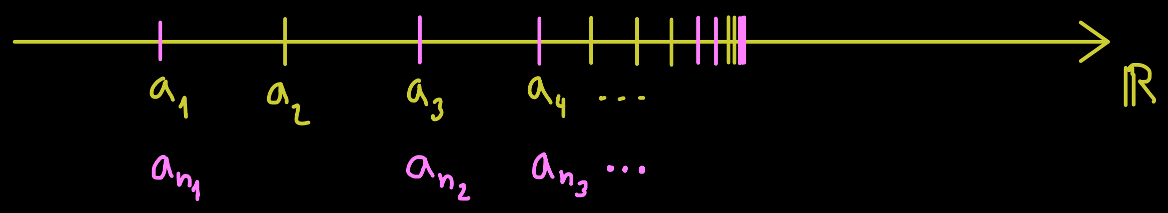
Real Analysis – Part 9



Definition: Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers that is strictly monotonically increasing, then $(a_{n_k})_{k\in\mathbb{N}}$ is called a subsequence of $(a_n)_{n\in\mathbb{N}}$. $(\forall k\in\mathbb{N}: \quad n_{k+1} > n_k)$

Example: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$,

$n = 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\ldots$

$$\left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}, \ldots\right)$$

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2^k})_{k\in\mathbb{N}} = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \ldots\right)$$

Fact: $(a_n)_{n\in\mathbb{N}}$ convergent with $\lim\limits_{n\to\infty} a_n = a$

$\Longrightarrow$ every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent $\lim\limits_{k\to\infty} a_{n_k} = a$

Example: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$

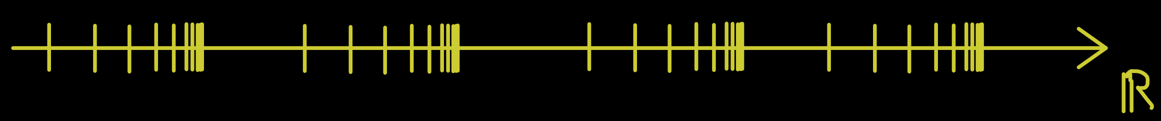
subsequence: $(a_{n_k})_{k\in\mathbb{N}} = (a_{2\cdot k})_{k\in\mathbb{N}} = (1, 1, 1, 1, 1, \ldots)$ limit 1

subsequence: $(a_{n_k})_{k\in\mathbb{N}} = (a_{2\cdot k+1})_{k\in\mathbb{N}} = (-1, -1, -1, \ldots)$ limit -1

accumulation values

Definition: $a\in\mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$

(cluster point, accumulation point, limit point, partial limit, ...)



Show: $a\in\mathbb{R}$ is an accumulation value of $(a_n)_{n\in\mathbb{N}}$

$\Longleftrightarrow$ $\forall \varepsilon > 0$: The ε-neighbourhood of a contains infinitely many sequence members of $(a_n)_{n\in\mathbb{N}}$

$= (a-\varepsilon, a+\varepsilon)$