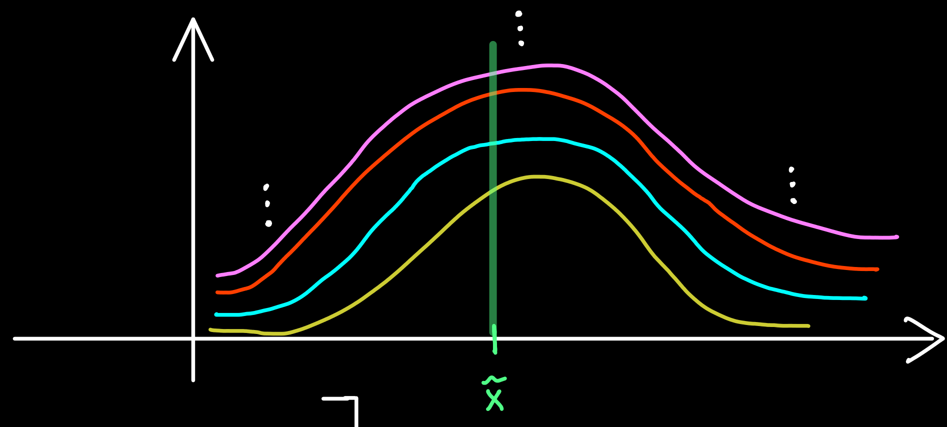
Real Analysis - Part 24

sequence of functions: $(f_1, f_2, f_3, f_4, f_5, \dots)$ $\quad f_n : I \to \mathbb{R}$

Pointwise convergence: $(f_1, f_2, f_3, f_4, f_5, \dots)$ is pointwisely convergent to a function $f : I \to \mathbb{R}$ if for all $\tilde{x} \in I$:

$$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), f_4(\tilde{x}), f_5(\tilde{x}), \dots)$$

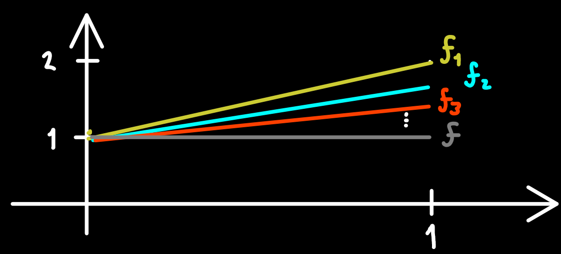
is convergent to $f(\tilde{x})$.



$$\left[\forall \tilde{x} \in I \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon \right]$$

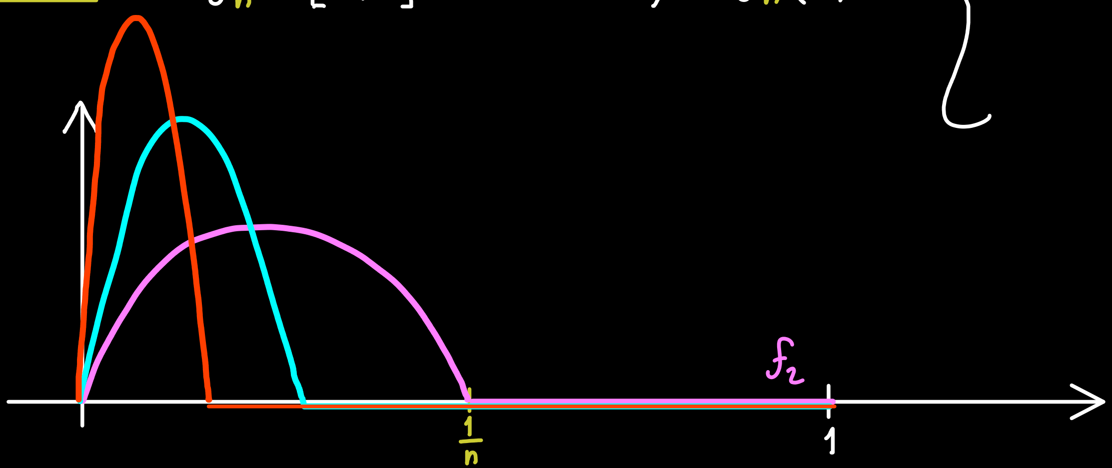
Example: $f_n : [0,1] \to \mathbb{R}\,,\quad f_n(x) = \frac{1}{n}x + 1$

For $\tilde{x} \in [0,1]$: $\quad f_n(\tilde{x}) = \frac{1}{n}\tilde{x} + 1 \xrightarrow{n \to \infty} 1$



$\Rightarrow$ (pointwise) limit function $\quad f : [0,1] \to \mathbb{R}\,,\quad f(x) = 1$

Example: $f_n : [0,1] \to \mathbb{R}\,,\quad f_n(x) = \begin{cases} n^2 x(1 - nx) & ,\ x \in [0, \frac{1}{n}] \\ 0 & ,\ x \in (\frac{1}{n}, 1] \end{cases}$



$$f_n\left(\tfrac{1}{2} \cdot \tfrac{1}{n}\right) = n^{\cancel{2}} \frac{1}{2\cancel{n}}\left(1 - \cancel{n}\frac{1}{2\cancel{n}}\right) = \frac{n}{4}$$

For $x = 0$: $\quad f_n(x) = 0 \quad$ for all $n \in \mathbb{N}$

For $x > 0$: $\quad f_n(x) = 0 \quad$ for all $n > \frac{1}{x}$

$\Rightarrow$ (pointwise) limit function $\quad f : [0,1] \to \mathbb{R}\,,\quad f(x) = 0$

Example: