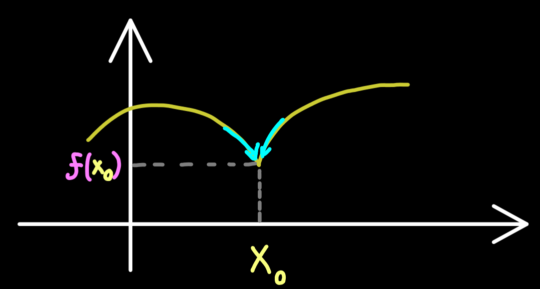
Real Analysis – Part 27

Definition: Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

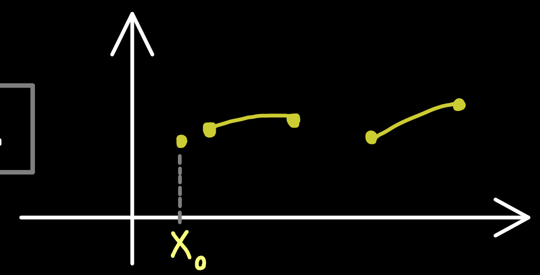
f is called continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$



or if x_0 is isolated in I.

There is no sequence $(x_n)_{n\in\mathbb{N}} \subseteq I \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$



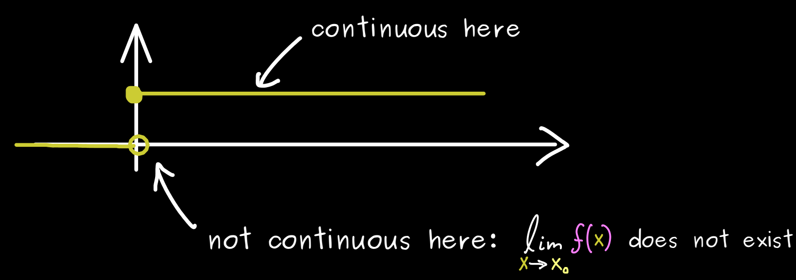
Definition: Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

f is called continuous (on I) if f is continuous at x_0 for all $x_0 \in I$.

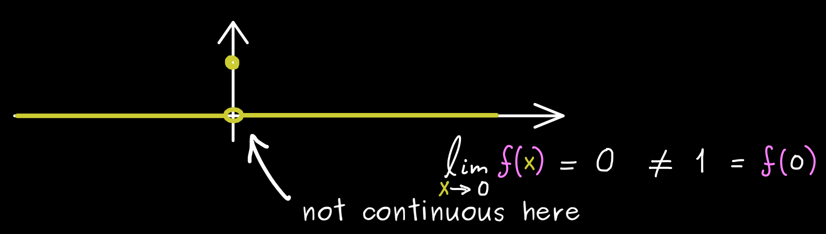
To remember: Continuity implies: $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$ (if $\lim_{n\to\infty} x_n \in I$)

Examples: (a) $f: I \to \mathbb{R}$ constant

(b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$



(c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

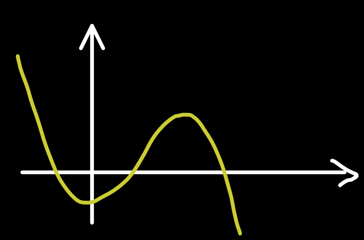


(d) $f: \mathbb{R} \to \mathbb{R}$ polynomial

$$f(x) = a_m \cdot x^m + a_{m-1} x^{m-1} + \cdots + a_1 \cdot x^1 + a_0$$

We have: $\lim_{x\to x_0} f(x) = f(x_0)$ for all $x_0 \in I$.

limit theorem for sequences



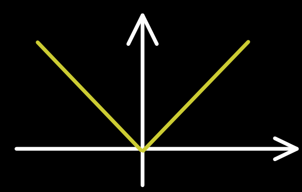
(e) $f: I \to \mathbb{R}$ rational function

$I := \{x \in \mathbb{R} \mid q(x) \neq 0\}$

$$f(x) = \frac{\overset{\text{polynomial}}{p(x)}}{\underset{\text{polynomial}}{q(x)}}$$ continuous on I

(f) $f: \mathbb{R} \to \mathbb{R}$ absolute value

$$f(x) = |x| = \begin{cases} -x , & x < 0 \\ x , & x \geq 0 \end{cases}$$

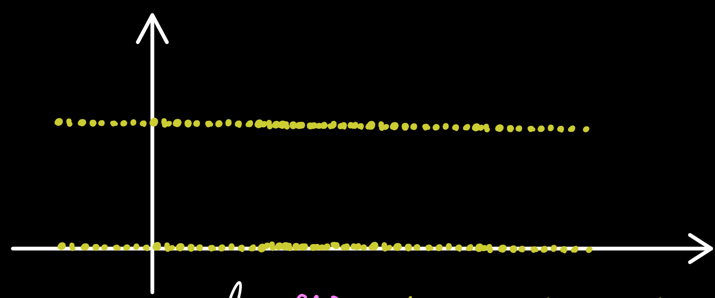


$$\left.\begin{aligned} \lim_{x \nearrow 0} f(x) &= \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (-x_n) = 0 \\ \lim_{x \searrow 0} f(x) &= \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (x_n) = 0 \end{aligned}\right\} \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

(g) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 , & x \notin \mathbb{Q} \\ 1 , & x \in \mathbb{Q} \end{cases}$$

($\mathbb{Q}$ is dense in $\mathbb{R}$ by construction)



$\lim_{x \to x_0} f(x)$ does not exist