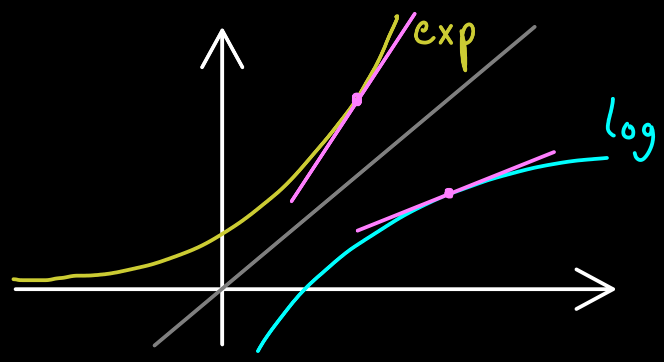
Real Analysis – Part 39

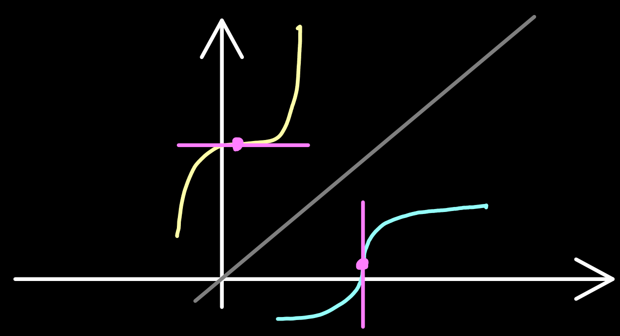
$\log : (0,\infty) \to \mathbb{R}$ defined by the inverse of $\underbrace{\exp : \mathbb{R} \to (0,\infty)}_{\text{differentiable}}$



Consider: $I, J \subseteq \mathbb{R}$ intervals, $f : I \to J$ bijective $\Rightarrow$ $f^{-1} : J \to I$ exists

Assume: f differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$

$y_0 := f(x_0)$



Choose sequence: $(y_n)_{n\in\mathbb{N}} \subseteq J \setminus \{y_0\}$ with $\lim\limits_{n\to\infty} y_n = y_0$

There is exactly one $x_n \in I$ with $f(x_n) = y_n$

$$\frac{f^{-1}(y_n) - f^{-1}(\underbrace{y_0}_{f(x_0)})}{y_n - y_0} = \frac{f^{-1}(f(x_n)) - f^{-1}(f(x_0))}{f(x_n) - f(x_0)} = \frac{x_n - x_0}{f(x_n) - f(x_0)}$$

$$= \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right)^{-1}$$

$$(f^{-1})'(y_0) = \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \left(\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \right)^{-1}$$

$$= \left(f'(x_0) \right)^{-1}$$

We need: $x_n \xrightarrow{n\to\infty} x_0$

$\Leftrightarrow$ $f^{-1}(y_n) \xrightarrow{n\to\infty} f^{-1}(y_0)$

$\Leftrightarrow$ f^{-1} continuous at y_0

Theorem: Let $I, J \subseteq \mathbb{R}$ be intervals and $f : I \to J$ be bijective.
If f is differentiable at x_0 with $f'(x_0) \neq 0$ and f^{-1} is continuous at $y_0 := f(x_0)$,
then f^{-1} is differentiable at y_0 with:

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

Example:

$$\log'(y) = \frac{1}{\exp'(\log(y))} = \frac{1}{\exp(\log(y))} = \frac{1}{y}$$